Modified similarity renormalization of Hamiltonians. Positronium on the light front *

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Abstract

Modified similarity renormalization (MSR) of Hamiltonians is proposed, that performes by means of flow equations the similarity transformation of Hamiltonian in the particle number space. This enables to renormalize in the energy space the field theoretical Hamiltonian and makes possible to work in a severe trancated Fock space for the renormalized Hamiltonian.

Original works of Wegner in solid state physics has served us as the guiding principle in constructing the MSR scheme.

The renormalized to the second order effective QED_{3+1} Hamiltonian on the light front is obtained. This Hamiltonian reproduces in $|e\bar{e}>$ sector the standard singlet-triplet splitting of positronium and recorves rotational symmetry of the canonical theory. The lowest and next-to-lowest energy states of positronium are almost independent on the cutoff when in both cases the 'same' (state and sector independent) counterterms are included.

The electron (photon) mass counterterms are IR (collinear) finite if all diagrams to the second order, arising from the flow equations and normal-ordering Hamiltonian, are taken into account, and vary with UV cutoff in accordance with 1-loop renormalization group equations.

Both approximations of perturbation theory and Fock space trancation are under the control and can be improved systematically within the proposed renormalization scheme.

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1 Introduction

About two years ago Glazek and Wilson proposed a new renormalization scheme for Hamiltonians, called similarity renormalization. The main idea of their approach is to transform the regularized bare Hamiltonian, with a large cutoff Λ, into an effective Hamiltonian that has no matrix elements with energy jumps larger compared to the small cutoff $\lambda < \Lambda$.

The initial field theoretical Hamiltonian contains usually fluctuations (or states) of all multiple energy scales, that couple each other. The effective Hamiltonian of Glazek and Wilson has by construction a band-diagonal form in the energy space. This means that the different energy scales present in the initial Hamiltonian are decoupled in the effective Hamiltonian. This is the main idea of renormalization group approach to take under the control the effects of high-energy states, that gives rise to the effective low-energy theory.

In a relativistic field theory the Hamiltonian usually contains interactions which change the particle number of a state. In the Hamiltonian matrix in Fock space representation many-body states couple to few-body states. This means that relativistic Hamiltonian is infinite in particle number space. The same is true for the renormalized in the energy space effective Hamiltonian of Glazek and Wilson as far as it contains the particle number changing interactions.

What kind of problems arise in solving the relativistic bound state problem by diagonalizing finite dimensional Hamiltonian matrices numerically. Suppose, that the energy renormalization is performed in the frame of Glazek,Wilson and we are left with the finite in energy space but infinite in particle number space effective Hamiltonian.

First the calculation of effective Hamiltonian so far done are always perturbative [6],[11]. This means that a kind of Tamm-Dancoff (TD) trancation, which restricts the total number of particles in any intermediate states, is performed. There are still effective induced interactions, together with the canonical one, which change the particle number by two in the band-diagonal effective Hamiltonian. To restrict the Fock space in this case is to throw away the interactions corresponding to the diagrams with more particles in the intermediate states. Therefore the narrow range of the cutoff (band) λ, $m\alpha^2 < \lambda < m\alpha$ for QED [6], is always introduced in Glazek,Wilson aproach, where the correct and cutoff independent values of physical masses are obtained. As far as the cutoff λ shifted below this range, the perturbative theory blows up and the contribution of higher orders in coupling constant diagrams is needed to cure the situation [6]. At least for the theories well defined in IR domain as QED we need to elaborate the systematic procedure that avoids these difficulties. As was mentiond above the source of these difficulties are the interactions which change the number of particles in a state.

Second, in Glazek,Wilson approach the elimination of far-off diagonal matrix elements is performed in each particle number sector, including the sectors that conserve the number of particles. In particle number conserving sector the interactions are induced, corresponding to the marginal relevant operators of the theory. The elimination of such terms can cause the convergency problems [3]. Within the framework of Wegner's flow equations, this has already been observed for the one-dimensional problem and solved by the proper choice of the generator of unitary transformation [1]. This has served us as a guiding principle in constructing the MSR scheme.

The source of all these difficulties is the presence of interactions in the effective Hamiltonian, which change the number of particles in a state and mix different Fock components. This causes also the general problems of the so-called 'sector-dependent' (occuring in a naive TD approximation [12]) and 'state-dependent' (occuring in the method of iterrative resolvents [15]) counterterms. Namely, it means,that the counterterms are needed to be added which depend on the Fock sector [12] (and physical state [15]) to get the cutoff independent physical masses

of states.

Therefore our aim is to modify the similarity scheme of Glazek, Wilson in a way to decouple different particle number states, that means to eliminate the interactions which change the number of particles in a state. Generally speaking, we aim to simulate the effects not only of high-energy states on the low-energy one, what conventional renormalization procedure does,but also the effects of many-body states on the few-body one by a set of effective interactions which do not change particle number and do not couple to high-energy states either. In this way we are able to control the approximations of the perturbation theory (or any other trancation made), and improve it systematically.

How does the 'modified similarity renormalization' (MSR) work? Say we have the two-body bare Hamiltonian, which changes the state of at most two particles (the case of light-front QED, QCD). The Hamiltonian is therefore the pentadiagonal block matrix in the representation of different number of particle-hole states. Also there is a large number,restricted by the UV-cutoff Λ, of states in the energy representation within each block fig.1, so that there are (infinitely) many 'energy states' and 'Fock components' in the field theoretical Hamiltonian.The idea of the MSR is to perform similarity transformation, analogous to similarity transformation in the 'energy space' of Glazek,Wilson, but in the 'particle number space'. This results in effective Hamiltonian with a block-diagonal structure in a 'particle number space'. How does it work technically? We apply the flow equations in a way to eliminate the far-off diagonal elements in the energy space (the matrix elements with large energy jumps $|E_i - E_j| > \lambda$) only for those blocks that change the number of particles. This is to be done continuously in a differential way for all cutoffs from λ equal to bare cutoff up to λ tending to zero. For the value of $\lambda = 0$ all interactions, canonical and induced, which change the number of particles are completly eliminated (except for diagonal in the 'energy space' matrix elements, that do not contribute in physical processes). One can say that instead of **band-diagonalization** in the 'energy space' in each 'particle number block' fig.2 (similarity renormalization (SR) of Glazek, Wilson) we perform **block-diagonalization** in 'particle number space', where the elimination of the blocks, which change the number of particles, is going in the 'energy space' fig.4 (modified similarity renormalization (MSR)).

Such construction of MSR brings several advantages with respect to the known methods to get the effective theories. First, we are able to treat different energy scales in sequence, what is usual for the renormalization approach. This solves the problem with divergencies, appearing when single unitary transformation (by one step) is performed and all energy scales are treated at once [12], [16]. Second,in MSR we hope to control both effects of high-energy and many-body states.

One can distiguish then two 'renormalizations', in 'energy' and 'particle number' space, that are closely related to each other in a complicated way.

What are the main differences between two approaches (SR and MSR) and consequences for the solution of bound state problem? First,in MSR there are still matrix elements with large energy jumps in particle number conserving sectors. Second, it is possible in MSR to eliminate **completly** the interactions which change the number of particles, i.e. the limit of $\lambda \to 0$ for the effective Hamiltonian is well defined (as $\lambda \to 0$ no patalogies occure in MSR in contrast to the case of band-diagonal effective Hamiltonian in SR).

We define the effective Hamiltonian

$$H_{eff}(\Lambda) = \lim_{\lambda \to 0} H_B(\lambda, \Lambda) \tag{1}$$

where $H_B(\lambda, \Lambda)$ is obtained by the unitary transform of the bare Hamiltonian

$$H_B(\lambda, \Lambda) = U(\lambda, \Lambda) H_B(\Lambda) U^+(\lambda, \Lambda) \tag{2}$$

The unitary transformation $U(\lambda, \Lambda)$ is specified in the flow equation method below. The renormalized effective Hamiltonian is defined

$$H_{ren} = \lim_{\Lambda \to \infty} H_{eff}(\Lambda) \tag{3}$$

Accepting the above definitions we obtain the renormalized effective Hamiltonian in the 'particle number conserving' sectors, which do not depend explicitly on the UV-cutoff. Implicit dependence on the cutoff due to the renormalization group running of coupling constants and masses is present. We leave aside in this work the question on the new types of counterterms,probably appeared due to the new generated interactions in effective Hamiltonian and which can not be absorbed by the running of relevant and marginal operators of the canonical theory [17]. This means that the physical results become insensitive to the cutoffs.

The only problem that can arise now by numerical diagonalization of the effective Hamiltonian matrix, that depend on the large UV-cutoff Λ as size of the matrix and does not depend on the artificial cutoff $\lambda < \Lambda$, is the neccesity to introduce the 'sector-dependent' [12] or 'state-dependent' [15] counterterms to get the cutoff independent physical masses of states. This problem is solved in the method of MSR, where the resulting effective Hamiltonian is diagonal in 'particle number space' and different particle number (Fock) states are completly decoupled. Explicitly, the lowest and next-to-lowest energy states are almost independent on the cutoff when in both cases the same counterterms are included. This is true despite that we perform the calculations perturbatively and work in a severly trancated space [10].

In this work we give the main points of MSR and consider QED on the light front to illustrate the method. The key ingredient of MSR are flow equations that perform infinitesimal unitary transformation in a continuous way [1]. On the other hand Wegner's flow equations can be used in SR scheme to diagonalize a given Hamiltonian in the energy space approximately. If in that case one performs the integration of the flow equations only to a finite value of the flow parameter, one obtains a Hamiltonian with a band-diagonal structure [8]. In MSR scheme we use the flow equations to transform the Hamiltonian into a block-diagonal form in the particle number space. Both approaches (SR and MSR) are usefull depending on the problem one wants to treat. The purpose of this paper is to illustrate the advantages of MSR manifest in solving the bound state problem.

2 Flow equations and MSR

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The flow equation is written

$$\frac{dH(l)}{dl} = [\eta(l), H(l)] \tag{4}$$

This is the differential form of the continuous unitary transformation acting on the Hamiltonian in general; $\eta(l)$ is the generator of the transformation and l is the flow parameter. The aim of the Wegner's flow equations is to bring the initial Hamiltonian matrix $H(l = 0)$, written in the energy space, to a diagonal (or block-diagonal where exact diagonalization is not possible) form. The finite values of the flow parameter l correspond to intermediate stages of diagonalization with the band-diagonal structure of Hamiltonian (SR) [8]. The result of the procedure at $l \to \infty$ is the 'approximately' diagonal Hamiltonian, where off-diagonal matrix elements do not vanish, but they become exponentially small. Also the transformation is designed in a way to avoid small energy denominators usually present in the perturbation theory.

What is the choice of the generator $\eta(l)$ that performs such transformation? Break the Hamiltonian into 'diagonal' and 'rest' parts $H = H_d + H_r$. Then the prescription of Wegner is [1]

$$\eta(l) = [H_d(l), H_r(l)] \tag{5}$$

At this step the unitary transformation is defined completly. The only freedom left is the princip of separation into 'diagonal' and 'rest' parts. It depends on the problem one wants to treat, how to separate the relevant information (degrees of freedom), carried by H_d, and irrelevant one, put in H_r and eliminated by the unitary transformation.

Our goal is to transform the Hamiltonian into the blocks with the same number of (quasi)particles (MSR). This means, that we define the 'diagonal' part H_d as conserving the number of particles part before and after interaction, and the 'rest' H_r as particle number changing part. In the case of QED(QCD), where the electron-photon (quark-gluon) coupling is present, the number of photons (gluons) is conserved in each block of the effective Hamiltonian.

As a result of unitary transformation the new interactions are induced (see further). They correspond to the marginal relevant operators of the theory, i.e. are irrelevant at $l = 0$ and manifest themself only at finite value of l. This means, that these terms must be added to the canonical Hamiltonian at $l \neq 0$, and therefore give rise to the new terms in the generator of transformation $\eta(l)$. This in its turn generates new interactions again.

To be able perform the calculations analytically we proceed further in perturbative frame and trancate this series assuming the coupling constant is small.

As illustration of the method we consider in the work QED on the light front, therefore we proceed further in application to it. For any finite value of l one has

$$H(l) = H_{0d} + H_r^{(1)} + H_d^{(2)} + H_r^{(2)} + ... \tag{6}$$

where the subscript above denotes the order in bare coupling constant, $H^{(n)} \sim e^n$; indices 'd'-diagonal, 'r'-rest parts correspondingly. The part H_{0d} is the free Hamiltonian, corresponding to the single particle energies with the structure in secondary quantization $a^+a, b^+b(d^+d)$; $H_r^{(1)}$ denotes electron-photon coupling (of the type a^+b^+b); $H_d^{(2)}$ is the second order diagonal part of the Hamiltonian, having the structure $b^+d^+bd, b^+b^+bb, d^+d^+dd$ (in the light front this corresponds to the canonical instantaneous (seegull) and new generated to the second order interactions in 'diagonal' sector).

Note, that the diagonal part in MSR scheme is not only the free Hamiltonian but the particle number conserving part of the effective Hamiltonian, which contains also corresponding interactions. The choice of only H_0 for the diagonal part give rise to the band-diagonal structure of the effective Hamiltonian in each Fock sector in SR scheme [8].

The generator of transformation is

$$\eta(l) = [H_d, H_r] = [H_{0d}, H_r^{(1)}] + [H_{0d}, H_r^{(2)}] + ... = \eta^{(1)} + \eta^{(2)} + ... \tag{7}$$

To the second order the flow equation is written

$$\frac{dH(l)}{dl} = [\eta, H] = [[H_{0d}, H_r^{(1)}]H_{0d}] + [[H_{0d}, H_r^{(1)}]H_r^{(1)}] + [[H_{0d}, H_r^{(2)}]H_{0d}] + ... \tag{8}$$

The new terms of higher orders e^n than present in the canonical Hamiltonian are generated by flow equations.

We choose for the basis the single particle states

$$H_{0d}|i>= E_i|i> \tag{9}$$

Then in matrix form one has

$$\eta_{ij} = (E_i - E_j)H^{(1)}_{rij} + (E_i - E_j)H^{(2)}_{rij} + ...$$
$$\frac{dH_{ij}}{dl} = -(E_i - E_j)^2 H^{(1)}_{rij} + [\eta^{(1)}, H^{(1)}_r]_{ij} - (E_i - E_j)^2 H^{(2)}_{rij} + ... \quad (10)$$

Neglecting the dependence of the single particle energies on the flow parameter $E_i(l)$, one has to the first order

$$\frac{dH^{(1)}_{rij}}{dl} = -(E_i - E_j)^2 H^{(1)}_{rij}$$
$$H^{(1)}_{rij}(l) = H^{(1)}_{rij}(l=0)\mathrm{e}^{-(E_i-E_j)^2 l} = H^{(1)}_{rij}(\lambda = \Lambda \to \infty)\mathrm{e}^{-\frac{(E_i-E_j)^2}{\lambda^2}} \quad (11)$$

Here we have used the physical sence of the flow parameter l. Namely, in SR scheme, it defines the size of the band λ, corresponding to the UV-cutoff, where the effective Hamiltonian is defined ($|E_i - E_j| < \lambda$) [8]. The connection between these two values is

$$l = \frac{1}{\lambda^2} \quad (12)$$

In the MSR scheme only the interactions, which change the number of particles, are defined in band size λ, while the particle number conserving part of the effective Hamiltonian exist everywhere (matrix elements with all energy differences are present).

For the following analyse it is useful to introduce the similarity function

$$f_{ij}(l) = e^{-(E_i-E_j)^2 l} = \mathrm{e}^{-\frac{(E_i-E_j)^2}{\lambda^2}} \quad (13)$$

which characterizes the behavior (fall off) of the leading order 'rest' interaction with the cutoff λ.

We define the initial Hamiltonian at the bare UV-cutoff $\Lambda \to \infty$ (corresponding to the flow parameter $l = 0$) and scale it by flow equations down to the cutoff $\lambda < \Lambda$. The resulting effective Hamiltonian is defined in the limit $\lambda \to 0$ ($l \to \infty$), where off-diagonal matrix elements of the particle number changing interactions are completly eliminated. It turns out that the diagonal elements of these interactions contribute the trivial terms to the 'diagonal' sectors in particle space (see further). Therefore one can assume that the effective Hamiltonian at $\lambda \to 0$ has only particle number conserving interactions, i.e. has the block-diagonal form in particle space.

To the second order one has to distinguish the 'diagonal' and 'rest' terms. For the 'rest' part one has

$$\frac{dH^{(2)}_{rij}}{dl} = [\eta^{(1)}, H^{(1)}]_{rij} - (E_i - E_j)^2 H^{(2)}_{rij} \quad (14)$$

where index 'r' by $[\eta^{(1)}, H^{(1)}]_r$ defines the particle number changing part of the commutator. Introduce

$$H^{(2)}_{rij}(l) = \mathrm{e}^{-(E_i-E_j)^2 l}\tilde{H}^{(2)}_{rij}(l) \quad (15)$$

i.e. the 'rest' part is defined in the energy band $\lambda = 1/\sqrt{l}$. Then the solution reads

$$\tilde{H}^{(2)}_{rij}(l) = \tilde{H}^{(2)}_{rij}(l=0) + \int_0^l dl' \mathrm{e}^{(E_i-E_j)^2 l'}[\eta^{(1)}, H^{(1)}]_{rij}(l') \quad (16)$$

For the 'diagonal' part one has

$$\frac{dH^{(2)}_{dij}}{dl} = [\eta^{(1)}, H^{(1)}]_{dij} \tag{17}$$

and the solution is

$$H^{(2)}_{dij}(l) = H^{(2)}_{dij}(l=0) + \int_0^l dl' [\eta^{(1)}, H^{(1)}]_{dij}(l') \tag{18}$$

Note, that though in general the commutator $[[H_{0d}, H^{(2)}_d]H_{0d}]$ is not zero, it is not present in the flow equation due to the definition of the diagonal part. The corresponding commutator $[[H_{0d}, H^{(2)}_r]H_{0d}]$ in the 'nondiagonal' sector insures the band-diagonal form for the 'rest' interaction and also gives rise to the different structure of generated interaction (the integral term) in 'rest' and 'diagonal' sectors.

The commutator $[\eta^{(1)}, H^{(1)}]$ gives rise to the new terms to the second order in bare coupling e. In the case of QED it induces the new types of interactions, corresponding to the marginal relevant operators of the theory, and generates the renormalization group corrections to the electron (photon) masses. The coupling constant starts to run to the third order in e.

Further we consider in 'diagonal' sector all contributions to the second order to protect gauge invariance, though in practical calculations we focus separately on the problem of new generated interactions and calculation of corresponding counterterms.

Note, that the MSR scheme (and also SR) enables to build the effective low energy Hamiltonian together with all, 'canonical' and possible to appear 'new', counterterms, found (from the coupling coherence condition [17],[4]) order by order in coupling constant e. This defines the renormalized Hamiltonian used for the solution of bound state problem numerically.

3 Renormalized effective Hamiltonian

Using the flow equations, we derive in this section the renormalized to the second order in bare coupling effective Hamiltonian for QED_{3+1} on the light front.

3.1 Canonical light-front QED_{3+1} Hamiltonian

We start with the canonical light-front QED Hamiltonian H_{can}, devided into free and interacting parts

$$P^- = H_{can} = \int dx^- d^2x^\perp (\mathcal{H}_0 + \mathcal{H}_I) \, . \tag{19}$$

In light-front gauge $A^+ = A^0 + A^3 = 0$, the constrained degrees of freedom A^- and ψ_- ($\psi = \psi_+ + \psi_-, \psi_\pm = \Lambda_\pm \psi$) can be removed explicitly; this gives the light-front gauge Hamiltonian defined through the independent physical fields $A_\perp$ and ψ_+ only [5]

$$\mathcal{H}_0 = \frac{1}{2}(\partial^i A^j)(\partial^i A^j) + \xi^+ \left(\frac{-\partial_\perp^2 + m^2}{i\partial^+} \right) \xi \, , \tag{20}$$

$$\mathcal{H}_I = \mathcal{H}_{ee\gamma} + \mathcal{H}_{ee\gamma\gamma} + \mathcal{H}_{eeee} \tag{21}$$

and

$$\mathcal{H}_{ee\gamma} = e\xi^+ \left[-2(\frac{\partial^\perp}{\partial^+} \cdot A^\perp) + \sigma \cdot A^\perp \frac{\sigma \cdot \partial^\perp + m}{\partial^+} + \frac{\sigma \cdot \partial^\perp + m}{\partial^+} \sigma \cdot A^\perp \right] \xi \, , \tag{22}$$

$$\mathcal{H}_{ee\gamma\gamma} = -ie^2 \left[\xi^+\sigma\cdot A^\perp \frac{1}{\partial^+}(\sigma\cdot A^\perp\xi)\right] \ , \tag{23}$$

$$\mathcal{H}_{eeee} = 2e^2 \left[\left(\frac{1}{\partial^+}(\xi^+\xi)\right)\left(\frac{1}{\partial^+}(\xi^+\xi)\right)\right] \ , \tag{24}$$

where $\{\sigma^i\}$ are the standard 2×2 Pauli matrices, and $\partial^+ = 2\partial_- = 2\frac{\partial}{\partial x^-}$. We have used the two-component representation for fermion fields introduced by Zhang and Harindranath [5] $\psi_+ = \binom{\xi}{0}$. To simplify the calculations we rewrite all interactions through creation and annihilation operators. This turns out to be useful in the flow equations formalism, [3].

Following standard quantum field theory procedure we use the momentum-space representation for the field operators, [4] and [5],

$$\begin{aligned} \xi(x) &= \sum_s \chi_s \int \frac{dp^+ d^2p^\perp}{2(2\pi)^3}\theta(p^+)(b_{p,s}e^{-ipx} + d_{p,\bar{s}}e^{ipx}) \\ A^i(x) &= \sum_\lambda \int \frac{dq^+ d^2q^\perp}{2(2\pi)^3}\frac{\theta(q^+)}{\sqrt{q^+}}(\varepsilon^i_\lambda a_{q,\lambda}e^{-iqx} + h.c.) \ , \end{aligned} \tag{25}$$

where spinors are $\chi^{tr}_{1/2} = (1,0)$, $\chi_{-1/2^{tr}} = (0,1)$, with $\bar{s} = -s$ and polarization vectors $\varepsilon^i_1 = \frac{-1}{\sqrt{2}}(1,i)$, $\varepsilon^i_{-1} = \frac{1}{\sqrt{2}}(1,-i)$; the integration running over the $p^+ \geq 0$ only these states, that are allowed the light-front theory.

The corresponding (anti)commutation relations are

$$\begin{aligned} \{b_{p,s}, b^+_{p',s'}\} = \{d_{p,s}, d^+_{p',s'}\} = \bar{\delta}_{p,p'}\delta_{ss'} \\ [a_{q,\lambda}, a^+_{q',\lambda'}] = \bar{\delta}_{q,q'}\delta_{\lambda,\lambda'} \qquad , \end{aligned} \tag{26}$$

where

$$\bar{\delta}_{p,p'} \equiv 2(2\pi)^3\delta(p^+ - p'^+)\delta^{(2)}(p^\perp - p'^\perp) \ . \tag{27}$$

The light-front vacuum has trivial structure for both boson and fermion sectors, namely $a_q|0>= 0$; $b_p|0>= 0$, simpifying the analitical calculations. The normalization of states is according to

$$< p_1, s_1|p_2, s_2 >= \bar{\delta}_{p_1,p_2}\delta_{s_1,s_2} \ , \tag{28}$$

where $b^+_{p,s}|0>= |p,s>$.

Making use of the field representation eq. (25), we have the following Fourier transformed for
the **free** Hamiltonian

$$H_0 = \sum_s \int \frac{dp^+ d^2p^\perp}{2(2\pi)^3}\theta(p^+)\frac{p^{\perp 2}+m^2}{p^+}(b^+_{p,s}b_{p,s} + d^+_{p,s}d_{p,s}) + \sum_\lambda \int \frac{dq^+ d^2q^\perp}{2(2\pi)^3}\theta(q^+)\frac{q^{\perp 2}}{q^+}a^+_{q,\lambda}a_{q,\lambda} \ , \tag{29}$$

the leading order $O(e)$-**the electron-photon coupling**

$$\begin{aligned} H_{ee\gamma} &= \sum_{\lambda s_1 s_2}\int_{p_1p_2q}[g^*_{p_1p_2q}(l)\varepsilon^i_\lambda \tilde{a}_q + g_{p_1p_2q}(l)\varepsilon^{i*}_\lambda \tilde{a}^+_{-q}](\tilde{b}^+_{p_2}\tilde{b}_{p_1} + \tilde{b}^+_{p_2}\tilde{d}^+_{-p_1} + \tilde{d}_{-p_2}\tilde{b}_{p_1} + \tilde{d}_{-p_2}\tilde{d}^+_{-p_1}) \\ &\quad \times\chi^+_{s_2}\Gamma^i_l(p_1,p_2,-q)\chi_{s_1}\bar{\delta}_{q,p_2-p_1} \ , \end{aligned} \tag{30}$$

where

$$\Gamma^i_l(p_1,p_2,q) = 2\frac{q^i}{q^+} - \frac{\sigma\cdot p_2^\perp - im}{p_2^+}\sigma^i - \sigma^i\frac{\sigma\cdot p_1^\perp + im}{p_1^+} \ , \tag{31}$$

The l-dependence of the vertex Γ^i_l arises from the dependence of light-front energies (masses) on the flow parameter. Further we have for the **instantaneous** interactions of the order $O(e^2)$

$$H^{inst}_{eeee} = \sum_{s_1s_2s_3s_4}\int_{p_1p_2p_3p_4} g^{eeee}_{p_1p_2p_3p_4}(l)(\tilde{b}^+_{p_3}+\tilde{d}_{-p_3})(\tilde{b}^+_{p_4}+\tilde{d}_{-p_4})(\tilde{b}_{p_1}+\tilde{d}^+_{-p_1})(\tilde{b}_{p_2}+\tilde{d}^+_{-p_2})$$
$$\times\chi^+_{s_3}\chi^+_{s_4}\frac{4}{(p^+_1-p^+_3)^2}\chi_{s_1}\chi_{s_2}\bar{\delta}_{p_3+p_4,p_1+p_2} \tag{32}$$

and

$$H^{inst}_{ee\gamma\gamma} = \sum_{s_1s_2\lambda_1\lambda_2}\int_{p_1p_2q_1q_2} g^{ee\gamma\gamma}_{p_1p_2q_1q_2}(l)(\varepsilon^{i*}_{\lambda_1}\tilde{a}^+_{q_1}+\varepsilon^i_{\lambda_1}\tilde{a}_{-q_1})(\varepsilon^j_{\lambda_2}\tilde{a}_{q_2}+\varepsilon^{j*}_{\lambda_2}\tilde{a}^+_{-q_2})(\tilde{b}^+_{p_2}+\tilde{d}_{-p_2})(\tilde{b}_{p_1}+\tilde{d}^+_{-p_1})$$
$$\times\chi^+_{s_2}\frac{\sigma^j\sigma^i}{(p^+_1-q^+_1)}\chi_{s_1}\bar{\delta}_{p_1+q_2,q_1+p_2}\ ; \tag{33}$$

here

$$\tilde{a}_q\equiv a_{q,\lambda}\frac{\theta(q^+)}{\sqrt{q^+}},\qquad \left[\tilde{a}_{-q}\equiv a_{-q,\lambda}\frac{\theta(-q^+)}{\sqrt{-q^+}}\right]\ ,$$
$$\tilde{b}_p\equiv b_{p,s}\theta(p^+),\qquad \tilde{d}_p\equiv d_{p,\bar{s}}\theta(p^+)\ , \tag{34}$$

and the $\bar{\delta}$-simbol stands for the function defined in eq. (27), the short notation for the integral is

$$\int_p\equiv\int\frac{dp^+d^2p^\perp}{2(2\pi)^3}\ . \tag{35}$$

In the formulas above we write explicitly the momentum dependence of the coupling constants as long as $l\neq 0$. The initial conditions for the couplings are defined at the value of the bare cutoff $\Lambda\to\infty(l_\Lambda=0)$, namely

$$\lim_{\Lambda\to\infty} g^{ee\gamma}(l_\Lambda)=e \tag{36}$$

and for both instantaneous interaction couplings

$$\lim_{\Lambda\to\infty} g^{inst}(l_\Lambda)=e^2\ ; \tag{37}$$

this corresponds to the couplings of the canonical theory.

3.2 Flow equations for QED_{3+1} Hamiltonian on the light front

3.2.1 Generated interaction in $|e\bar{e}>$ sector

Following the procedure outlined in the second section, the leading order generator of the unitary transformation is

$$\eta^{(1)}(l) = \sum_{\lambda s_1s_2}\int_{p_1p_2q}(\eta^*_{p_ip_f}(l)\varepsilon^i_\lambda\tilde{a}_q+\eta_{p_ip_f}(l)\varepsilon^{i*}_\lambda\tilde{a}^+_{-q})(\tilde{b}^+_{p_2}\tilde{b}_{p_1}+\tilde{b}^+_{p_2}\tilde{d}^+_{-p_1}+\tilde{d}_{-p_2}\tilde{b}_{p_1}+\tilde{d}_{-p_2}\tilde{d}^+_{-p_1})$$
$$\times\chi^+_{s_2}\Gamma^i_l(p_1,p_2,-q)\chi_{s_1}\bar{\delta}_{q,p_2-p_1}\ , \tag{38}$$

$$\eta_{p_ip_f}(l)=-\Delta_{p_ip_f}g_{p_ip_f}=\frac{1}{\Delta_{p_ip_f}}\cdot\frac{dg_{p_ip_f}}{dl}\ . \tag{39}$$

where p_i and p_f stand for the set of initial and final momenta, respectively, and $\Delta_{p_ip_f} = \sum p_i^- - \sum p_f^-$, and the light-front fermion energy is $p^- = \frac{p^{\perp 2}+m^2}{p^+}$, the photon one $q^- = \frac{q^{\perp 2}}{q^+}$. In previous notations $g_{p_ip_f} = g_{p_1p_2q}$. Further we calculate the bound states of positronium. In what follows we consider in $|e\bar{e}>$ sector
the **generated interaction** to the first nonvanishing order

$$H^{gen}_{e\bar{e}e\bar{e}} = \sum_{s_1\bar{s}_2s_3\bar{s}_4}\int_{p_1p_2p_3p_4} V^{gen}_{p_ip_f}(l) b^+_{p_3}d^+_{p_4}d_{p_2}b_{p_1}\chi^+_{s_3}\chi^+_{\bar{s}_4}\chi_{\bar{s}_2}\chi_{s_1}\bar{\delta}_{p_1+p_2,p_3+p_4}\ , \tag{40}$$

with the initial condition $\lim_{\Lambda\to\infty} V^{gen}_{p_ip_f}(l_\Lambda) = 0$,
and the **instantaneous interaction**

$$H^{inst}_{e\bar{e}e\bar{e}} = \sum_{s_1\bar{s}_2s_3\bar{s}_4}\int_{p_1p_2p_3p_4} V^{inst}_{p_ip_f}(l) b^+_{p_3}d^+_{p_4}d_{p_2}b_{p_1}\chi^+_{s_3}\chi^+_{\bar{s}_4}\chi_{\bar{s}_2}\chi_{s_1}\bar{\delta}_{p_1+p_2,p_3+p_4}\ , \tag{41}$$

where

$$\begin{aligned} V^{inst}_{p_ip_f}(l) &= g^{inst}_{p_ip_f}(l)\,\frac{4}{(p_1^+-p_3^+)^2} \\ \lim_{\Lambda\to\infty} g^{inst}_{p_ip_f}(l_\Lambda) &= e^2\ , \end{aligned} \tag{42}$$

The order of the field operators in both interactions satisfies the prescription of standard Feynmann rules in the $|e\bar{e}>$ sector.

To the leading order we neglect the l dependence of light-front energies in the interactions, that enables to write the flow equations for the corresponding couplings.

The flow equations to the first (for the electron-photon coupling) and second (for the instantaneous and generated interactions) orders are

$$\begin{aligned} \frac{dg_{p_ip_f}(l)}{dl} &= -\Delta^2_{p_ip_f} g_{p_ip_f}(l) \\ \frac{dg^{inst}_{p_ip_f}(l)}{dl} &= 0 \\ \frac{dV^{gen}_{p_ip_f}(l)}{dl} &= <[\eta^{(1)}(l), H_{ee\gamma}]>_{|e\bar{e}>}\ , \end{aligned} \tag{43}$$

where

$$\Delta_{p_ip_f} = \sum p_i^- - \sum p_f^- \tag{44}$$

The matrix element $<[\eta^{(1)}(l), H_{ee\gamma}]>_{|e\bar{e}>}$ is understood as the corresponding commutator between the free electron-positron states, namely $<p_3s_3, p_4\bar{s}_4|...|p_1s_1, p_2\bar{s}_2>$.

Note, that the instantaneous and generated interactions are changing with the flow parameter l according to the flow equations in the 'particle number conserving' sector.

Renormalization group running of both (instantaneous and generated) interactions starts to the order $O(e^4)$, and the electron-photon coupling starts to run to the order $O(e^3)$.

Neglecting the dependence of the light-front energies on the flow parameter l, the solution reads

$$\begin{aligned} g_{p_ip_f}(l) &= f_{p_ip_f}\cdot e + O(e^3) \\ g^{inst}_{p_ip_f}(l) &= g^{inst}_{p_ip_f}(l_\Lambda = 0) = e^2 + O(e^4) \\ V^{gen}_{p_ip_f}(l) &= \int_0^l dl' <[\eta^{(1)}(l'), H_{ee\gamma}(l')]>_{|e\bar{e}>} + O(e^4) \\ f_{p_ip_f} &= \mathrm{e}^{-\Delta^2_{p_ip_f}l} = \mathrm{e}^{-\frac{\Delta^2_{p_ip_f}}{\lambda^2}}\ , \end{aligned} \tag{45}$$

where the subscript $|e\bar{e}>$ means, that the commutator is considered in the electron-positron sector. The electron-photon interaction exists in the band of size λ ($|\Delta_{p_i p_f}| < \lambda$), whereas the matrix elements of instantaneous and generated interactions in $|e\bar{e}>$ sector are defined for all energy differences.

We give further the explicit expressions for the generated interaction, and details of calculations can be found in Appendix B. In what follows we use the notations of this Appendix.

The matrix elements of the commutator $[\eta^{(1)}, H_{ee\gamma}]$ in the exchange and annihilation channels are

$$< [\eta^{(1)}, H_{ee\gamma}] > /\delta_{p_1+p_2,p_3+p_4} = \begin{cases} M_{2ii}^{(ex)} \frac{1}{(p_1^+ - p_3^+)} (\eta_{p_1,p_3} g_{p_4,p_2} + \eta_{p_4,p_2} g_{p_1,p_3}) \ , \\ \\ -M_{2ii}^{(an)} \frac{1}{(p_1^+ + p_2^+)} (\eta_{p_1,-p_2} g_{p_4,-p_3} + \eta_{p_4,-p_3} g_{p_1,-p_2}) \ , \end{cases} \tag{46}$$

where

$$\begin{aligned} \eta_{p_1,p_2}(l) &= e \cdot \frac{1}{\Delta_{p_1 p_2}} \frac{df_{p_1,p_2}(l)}{dl} \\ g_{p_1,p_2}(l) &= e \cdot f_{p_1,p_2}(l) \end{aligned} \tag{47}$$

and $\Delta_{p_1,p_2} = p_1^- - p_2^- - (p_1 - p_2)^-$. The matrix elements M_{2ii} between the corresponding spinors in both channels are

$$\begin{aligned} M_{2ij}^{(ex)} &= [\chi_{s_3}^+ \Gamma_l^i(p_1, p_3, p_1 - p_3) \chi_{s_1}] \, [\chi_{\bar{s}_2}^+ \Gamma_l^j(-p_4, -p_2, -(p_1 - p_3)) \chi_{\bar{s}_4}] \\ M_{2ij}^{(an)} &= [\chi_{s_3}^+ \Gamma_l^i(-p_4, p_3, -(p_1 + p_2)) \chi_{\bar{s}_4}] \, [\chi_{\bar{s}_2}^+ \Gamma_l^j(p_1, -p_2, p_1 + p_2) \chi_{s_1}] \end{aligned} \tag{48}$$

that determine the spin structure of the generated interaction.

We plug the formulas for commutator $[\eta^{(1)}, H_{ee\gamma}]$ together with the generator $\eta(l)$ and coupling constant $g(l)$, expressed through the similarity function $f(l)$, into the formula for generated interaction. This gives rise in both channels

$$\begin{aligned} V_{gen}^{(ex)}(\lambda) &= -e^2 M_{2ii}^{(ex)} \frac{1}{(p_1^+ - p_3^+)} \left(\frac{\int_\lambda^\infty \frac{df_{p_1,p_3,\lambda'}}{d\lambda'} f_{p_4,p_2,\lambda'} d\lambda'}{\Delta_{p_1,p_3}} + \frac{\int_\lambda^\infty \frac{df_{p_4,p_2,\lambda'}}{d\lambda'} f_{p_1,p_3,\lambda'} d\lambda'}{\Delta_{p_4,p_2}} \right) \\ V_{gen}^{(an)}(\lambda) &= e^2 M_{2ii}^{(an)} \frac{1}{(p_1^+ + p_2^+)} \left(\frac{\int_\lambda^\infty \frac{df_{p_1,-p_2,\lambda'}}{d\lambda'} f_{p_4,-p_3,\lambda'} d\lambda'}{\Delta_{p_1,-p_2}} + \frac{\int_\lambda^\infty \frac{df_{p_4,-p_3,\lambda'}}{d\lambda'} f_{p_1,-p_2,\lambda'} d\lambda'}{\Delta_{p_4,-p_3}} \right) . \end{aligned} \tag{49}$$

where in the integral we have neglected the dependence of light-front energies on the cutoff λ (that is the correction of order $O(e^2)$), and the connection between flow parameter and cutoff ($l = 1/\lambda^2$) is used.

Other unitary transformations can be performed, that aim to bring the field theoretical Hamiltonian to the block-diagonal form in 'particle number space'. The transformations we discuss further, performed in the frame of MSR (section 3.3), act also in the energy space and differ from the flow equations mainly of how fast the 'particle number changing' interactions are eliminated, i.e. by the convergency of the procedure.

Here we note, that the form of second order generated interaction, induced in MSR and written through the similarity functions $f_{p_i p_f}$, is universal for all transformations, which will be

discussed. Specifying the similarity function we obtain the explicit form of generated interaction, induced by different unitary transformations (section 3.3).

Following the flow equation presciption we use here

$$f_{p_1,p_2,\lambda} = \mathrm{e}^{-\frac{\Delta^2_{p_1p_2}}{\lambda^2}} \ . \tag{50}$$

that gives for the **generated interaction** in both channels fig.5

$$\begin{aligned} V^{(ex)}_{gen}(\lambda) \ &= \ -e^2 M^{(ex)}_{2ii} \frac{1}{(p^+_1 - p^+_3)} \frac{\Delta_{p_1,p_3} + \Delta_{p_4,p_2}}{\Delta^2_{p_1,p_3} + \Delta^2_{p_4,p_2}} \cdot (1 - f_{p_1,p_3,\lambda} f_{p_4,p_2,\lambda}) \\ V^{(an)}_{gen}(\lambda) \ &= \ e^2 M^{(an)}_{2ii} \frac{1}{(p^+_1 + p^+_2)} \frac{\Delta_{p_1,-p_2} + \Delta_{p_4,-p_3}}{\Delta^2_{p_1,-p_2} + \Delta^2_{p_4,-p_3}} \cdot (1 - f_{p_1,-p_2,\lambda} f_{p_4,-p_3,\lambda}) \ , \end{aligned} \tag{51}$$

By definition, given in the introduction, the renormalized effective Hamiltonian is obtained in the limit of cutoff tending to zero ($\lambda \to 0$). In this limit the electron-photon coupling, present in generated interaction through the similarity functions $f_{p_ip_f}$, is completly eliminated $f_{p_ip_f}(\lambda \to 0)$ for $\Delta_{p_ip_f} \neq 0$, and generated interaction is given by corresponding to 1 in bracket expression that does not depend explicitly on the cutoff λ. The implicit dependence as renormalization group running of coupling constant e and light-front energies (masses) to the next orders is present.

The modified similarity transformation is constructed to avoid divergencies in the form of small energy denominators, present in the second order perturbative approach. Namely, one has for the generated interaction in the exchange channel in the case of energy conserving process ($\Delta_{p_ip_f} = 0$)

$$\frac{\Delta_{p_1,p_3} + \Delta_{p_4,p_2}}{\Delta^2_{p_1,p_3} + \Delta^2_{p_4,p_2}} = \frac{\Delta_{p_1,p_3} + \Delta_{p_4,p_2}}{\Delta_{p_ip_f} + 2\Delta_{p_1,p_3}\Delta_{p_4,p_2}} \to \frac{1}{2}\left(\frac{1}{\Delta_{p_1,p_3}} + \frac{1}{\Delta_{p_4,p_2}}\right) \tag{52}$$

where

$$\Delta_{p_ip_f} \equiv p^-_1 + p^-_2 - p^-_3 - p^-_4 = \Delta_{p_1,p_3} - \Delta_{p_4,p_2} = \Delta_{p_1,-p_2} - \Delta_{p_4,-p_3} \tag{53}$$

due to the total momentum conservation in $'+'$ and 'transversal' directions. The divergent contribution in the generated interaction when $\Delta_{p_1,p_3} \sim \Delta_{p_4,p_2} \sim 0$ is effectively cancelled by the factor in bracket containing similarity functions $(1 - f_{p_1,p_3,\lambda} f_{p_4,p_2,\lambda})$.

Also it is obvious from eq. (52), that any energy differences (i.e. $\forall \Delta_{p_ip_f}$) are permitted for matrix elements of generated interaction in 'diagonal' sector.

For the further analyses we write the effective Hamiltonian at the finite value of λ, performing at the end the limit $\lambda \to 0$.

We rewrite the generated to the second order interaction in the form fig.8

$$\begin{aligned} \tilde{V}^{(ex)}_{gen}(\lambda) \ &= \ -e^2 N_{1,\lambda} \frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\tilde{\Delta}^2_1 + \tilde{\Delta}^2_2} \cdot \left(1 - \mathrm{e}^{-\frac{\Delta^2_1 + \Delta^2_2}{\lambda^4}}\right) \\ \tilde{V}^{(an)}_{gen}(\lambda) \ &= \ e^2 N_{2,\lambda} \frac{M^2_0 + M'^2_0}{M^4_0 + M'^4_0} \cdot \left(1 - \mathrm{e}^{-\frac{M^4_0 + M'^4_0}{\lambda^4}}\right) \ , \end{aligned} \tag{54}$$

where we have introduced

$$P^{+2}M^{(ex)}_{2ii,\lambda} = -N_1 \quad ; \qquad P^{+2}M^{(an)}_{2ii,\lambda} = N_2$$

$$\Delta_{p_1p_3} = \frac{\Delta_1}{P^+} = \frac{\widetilde{\Delta}_1}{(x'-x)P^+} \quad ; \qquad \Delta_{p_4p_2} = \frac{\Delta_2}{P^+} = \frac{\widetilde{\Delta}_2}{(x'-x)P^+};$$

$$\Delta_{p_1,-p_2} = \frac{M_0^2}{P^+} \quad ; \qquad \Delta_{p_4,-p_3} = \frac{M'^2_0}{P^+} \tag{55}$$

(see Appendix B for the explicit definition of these quantities in the light-front frame).

The expression eq. (55) is written for the rescaled value of the potential, i.e. $V_\lambda = \tilde{V}_\lambda/P^{+2}$, and the cutoff is defined in units of the total momentum P^+, i.e. $\lambda \to \frac{\lambda^2}{P^+}$, with $l = 1/\lambda^2$. The spin structure of the interaction is carried by the matrix elements M_{2ii}, defined in Appendix B.

We summarize the **instantaneous interaction** in both channels to the order $\mathbf{O(e^2)}$ fig. (5)

$$V^{(ex)}_{inst}(\lambda) = -\frac{4e^2}{(p_1^+ - p_3^+)^2}\ \delta_{s_1s_3}\delta_{s_2s_4}$$

$$V^{(an)}_{inst} = \frac{4e^2}{(p_1^+ + p_2^+)^2}\ \delta_{s_1\bar{s}_2}\delta_{s_3\bar{s}_4}\ , \tag{56}$$

where we have used $\chi^+_{s_3}\chi^+_{\bar{s}_2}\mathbb{1}\chi_{s_1}\chi_{\bar{s}_4} = \delta_{s_1s_3}\delta_{s_2s_4} + \delta_{s_1\bar{s}_2}\delta_{s_3\bar{s}_4}$. For the rescaled potential in the light-front frame Appendix B eq. (261) we have

$$\tilde{V}^{(ex)}_{inst,\lambda} = -\frac{4e^2}{(x-x')^2}\ \delta_{s_1s_3}\delta_{s_2s_4}$$

$$\tilde{V}^{(an)}_{inst,\lambda} = 4e^2\ \delta_{s_1\bar{s}_2}\delta_{s_3\bar{s}_4}\ . \tag{57}$$

Further we use the generated and instantaneous interactions in $|e\bar{e}>$ sector, obtained in this section, to calculate positronium bound state spectrum.

3.2.2 Renormalization issues

As was discussed above the commutator $[\eta^{(1)}, H_{ee\gamma}]$ also contributes to the self-energy term, giving rise to the renormalization of fermion and photon masses to the second order. The flow equation for the electron (photon) light-cone energy p^- is

$$\frac{dp^-}{dl} =< [\eta^{(1)}, H_{ee\gamma}] >_{self\ energy}\ , \tag{58}$$

where the matrix element is calculated between the single electron (photon) states $< p', s'|...|p, s >$. We drop the finite part and define $\delta p^-_\lambda = p^-(l_\lambda) - < |H_0| >$. Integration over the finite range gives

$$\delta p^-_\lambda - \delta p^-_\Lambda = \int_{l_\Lambda}^{l_\lambda} < [\eta^{(1)}, H_{ee\gamma}] >_{self\ energy} dl' = -\frac{(\delta\Sigma_\lambda(p) - \delta\Sigma_\Lambda(p))}{p^+}\ , \tag{59}$$

that defines the cutoff dependent self energy $\delta\Sigma_\lambda(p)$. The mass correction and wave function renormalization constant are given correspondingly, cf. [5]

$$\begin{aligned} \delta m_\lambda^2 &= p^+\delta p^-\Big|_{p^2=m^2} = -\delta\Sigma_\lambda(m^2) \\ Z_2 &= 1+\frac{\partial\delta p^-}{\partial p^-}\Big|_{p^2=m^2} \ . \end{aligned} \tag{60}$$

The on-mass-shell condition is defined through the mass m in the free Hamiltonian H_0.

We show further, that to the second order $O(e^2)$ the electron and photon masses and corresponding wave function renormalization constants in the renormalized Hamiltonian vary in accordance with the result of 1-loop renormalization group equations. This can serve as evidence for the equivalence of the flow equations and Wilson's renormalization. Therefore we have rewritten the mass correction δm_λ^2 through the self energy term, arising in 1-loop calculations of ordinary perturbative theory. The negative overall sign stems from our definition of the flow parameter, namely for $\Delta l > 0$ we are lowering the cutoff $dl = -\frac{2}{\lambda^3}d\lambda$.

We start with the bare cutoff mass $m_\Lambda^2 = m^2 + \delta M_\Lambda^{(2)}$, where $\delta M_\Lambda^{(2)}$ is the second order mass counterterm. According to eq. (59),eq. (60) the electron (photon) mass runs

$$m_\lambda^2 = m_\Lambda^2 - [\delta\Sigma_\lambda(m^2) - \delta\Sigma_\Lambda(m^2)] \tag{61}$$

defining, due to renormalizability, the counterterm $\delta M_\Lambda^{(2)} = \delta m_\Lambda^2 = -\delta\Sigma_\Lambda(m^2)$ and the dependence of the renormalized mass on the cutoff λ

$$m_\lambda^2 = m^2 + \delta m_\lambda^2 = m^2 - \delta\Sigma_\lambda \ . \tag{62}$$

We calculate explicitly the self-energy term. The **electron** energy correction contains several terms

$$\delta p_\lambda^- =< p', s'|H - H_0|p, s >= \left(\sum_{n=1}^{3}\delta p_{\lambda n}^-\right)\cdot\delta^{(3)}(p-p')\delta_{ss'} \ . \tag{63}$$

The first term is induced by the flow equation in single electron sector, namely comes from the commutator $[\eta(1), H_{ee\gamma}]$

$$\delta p_{1\lambda}^- = -\int_{l_\lambda}^{\infty} < [\eta^{(1)}, H_{ee\gamma}] >_{self\ energy} dl' = -\frac{\delta\Sigma_{1\lambda}(p)}{p^+} \ ; \tag{64}$$

it reads, cf. eq. (280) in AppendixC,

$$\begin{aligned} \delta p_{1\lambda}^- &= e^2\int\frac{d^2k^\perp dk^+}{2(2\pi)^3}\frac{\theta(k^+)}{k^+}\theta(p^+-k^+) \\ &\quad\times\Gamma^i(p-k,p,-k)\Gamma^i(p,p-k,k)\,\frac{1}{p^- - k^- - (p-k)^-}\times(-R) \ . \end{aligned} \tag{65}$$

This term explicitly depends on the cutoff λ ($l = 1/\lambda^2$) through the similarity function, that plays the role of a regulator in the loop integration

$$R_\lambda = f_{p,k,\lambda}^2 = \exp\left\{-2\left(\frac{\Delta_{p,k}}{\lambda}\right)^2\right\} \ . \tag{66}$$

Eq. (65) corresponds to the first diagram in fig. (6).

Two instantaneous diagrams, the second and third in fig. (6), contribute the cutoff independent (constant) terms. They arise from normal-ordered instantaneous interactions in the single electron sector and can be written as

$$\delta p_{n\lambda} = \delta p_n(l=0) = c_n < \hat{O}\hat{O}^+ > V_n^{inst}(l=0) \tag{67}$$

where $n = 2, 3$ corresponds to the second and third diagrams in fig. (6), c_n is the symmetry factor, $< \hat{O}\hat{O}^+ >$ stands for the boson ($n = 2$) and fermion ($n = 3$) contractions (i.e. $< \tilde{a}_k \tilde{a}_k^+ >= \theta(k^+)/k^+$ and $< \tilde{b}_p \tilde{b}_p^+ >= \theta(p^+)$), and $V_n^{inst}(l=0)$ arises from normal-ordering of $H_{ee\gamma\gamma}$ for $n = 2$ and of H_{eeee} for $n = 3$ (eqs. (32) and (33)).

These two diagrams $\delta p_n(l=0)$ define together with the first one $\delta p_1(l=0)$ the initial condition for the total energy correction, eq. (63).

Since the diagrams $n = 2, 3$ come from the normal-ordering canonical Hamiltonian at $l = 0$, they must accompany the first diagram for any flow parameter l. In what follows we use for the instantaneous terms the same regulator R, eq. (66)

$$\begin{aligned}
\delta p_{2\lambda}^- &= e^2 \int \frac{d^2k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+} \frac{\sigma^i\sigma^i}{[p^+ - k^+]} \times (-R) \\
\delta p_{3\lambda}^- &= e^2 \int \frac{d^2k^\perp dk^+}{2(2\pi)^3} \theta(k^+) \frac{1}{2} \left(\frac{1}{[p^+ - k^+]^2} - \frac{1}{[p^+ + k^+]^2} \right) \times (-R) \, .
\end{aligned} \tag{68}$$

We define the set of coordinates

$$\begin{aligned}
x &= \frac{k^+}{p^+} \\
k &= (xp^+, xp^\perp + \kappa^\perp) \, ,
\end{aligned} \tag{69}$$

where $p = (p^+, p^\perp)$ is the external electron momentum. Then the electron self energy diagrams, fig. (6), eq. (284) in AppendixC, contribute

$$\begin{aligned}
p^+\delta p_{1\lambda}^- &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \\
&\qquad \times \left[\frac{p^2 - m^2}{\kappa_\perp^2 + f(x)} \left(\frac{2}{[x]} - 2 + x \right) - \frac{2m^2}{\kappa_\perp^2 + f(x)} + \left(\frac{2}{[x]^2} + \frac{1}{[1-x]} \right) \right] \times (-R) \\
f(x) &= xm^2 - x(1-x)p^2
\end{aligned} \tag{70}$$

and

$$\begin{aligned}
p^+\delta p_{2\lambda}^- &= \frac{e^2}{8\pi^2} \int_0^\infty dx \int d\kappa_\perp^2 \left(\frac{1}{[x][1-x]} \right) \times (-R) \\
&\to \frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \left(\frac{1}{[x]} \right) \times (-R) \\
p^+\delta p_{3\lambda}^- &= \frac{e^2}{8\pi^2} \int_0^\infty dx \int d\kappa_\perp^2 \left(\frac{1}{[1-x]^2} - \frac{1}{(1+x)^2} \right) \times (-R) \\
&\to \frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa_\perp^2 \left(\frac{2}{[x]^2} \right) \times (-R) \, ;
\end{aligned} \tag{71}$$

for details we refer to AppendixC. Note, that the transformation in the integrals over x is performed before the regulator is taken into account [5]. (In the second integral the electron

momentum is replaced by the gluon one due to momentum conservation). The brackets '[ ]' denote the principle value prescription, defined further in eq. (77).

The loop integral over k eqs. (70) and (71) contains two types of divergencies: UV in the transversal coordinate $\kappa^\perp$ and IR in the longitudinal component k^+. The physical value of mass must be IR-finite. We show, that the three relevant diagrams together in fact give an IR-finite value for the renormalized mass; this enables to determine counterterms independent of longitudinal momentum. In the wave function renormalization constant, however, the IR-singularity is still present.

Define

$$\delta_1 = \frac{p^+}{P^+} \,, \tag{72}$$

where $P = (P^+, P^\perp)$ is the positronium momentum, p the electron momentum. The transversal UV divergency is regularized through the unitary transformation done, i.e. by the regulator R, eq. (66)

$$R_\lambda = \exp\left\{ -\left(\frac{\tilde{\Delta}_{p,k}}{\lambda^2 \delta_1} \right)^2 \right\} \;\approx\; \theta(\lambda^2\delta_1 - |\tilde{\Delta}_{p,k}|) \,, \tag{73}$$

where the cutoff is rescaled and defined in units of the positronium momentum P^+, namely $\lambda \to \sqrt{2}\lambda^2/P^+$, and $\Delta_{p,k} = p^- - k^- - (p-k)^- = \tilde{\Delta}_{p,k}/p^+$. The rude approximation for the exponential through a θ-function changes the numerical coefficient within a few percent; nevertheless it is useful to estimate the integrals in eqs. (70) and (71) in this way analitically. From eq. (73) we have for the sum of intermediate (electron and photon) state momenta (the external electron is on-mass-shell $p^2 = m^2$)

$$\frac{\kappa^{\perp 2}}{[x]} + \frac{\kappa^{\perp 2} + m^2}{[1-x]} \;\le\; \lambda^2\delta_1 + m^2 \tag{74}$$

giving for the regulator

$$\begin{aligned} R_\lambda &= \theta(\kappa^{\perp 2}_{\lambda max} - \kappa^{\perp 2})\, \theta(\kappa^{\perp 2}_{\lambda max}) \\ \kappa^{\perp 2}_{\lambda max} &= x(1-x)\lambda^2\delta_1 - x^2 m^2 \end{aligned} \tag{75}$$

and $\theta(\kappa^{\perp 2}_{\lambda max})$ leads to the additional condition for the longitudinal momentum

$$\begin{aligned} &0 \le x \le x_{max} \\ &x_{max} = \frac{1}{1 + m^2/(\lambda^2\delta_1)} \end{aligned} \tag{76}$$

implying that the singularity of the photon longitudinal momentum for $x \to 1$ is regularized by the function R_λ. This is the case due to the nonzero fermion mass present in eq. (74) for the intermediate state with $(1-x)$ longitudinal momentum. The IR-singularity when $x \to 0$ is still present; it is treated by the principle value prescription [5]

$$\frac{1}{[k^+]} = \frac{1}{2}\left(\frac{1}{k^+ + i\varepsilon P^+} + \frac{1}{k^+ - i\varepsilon P^+} \right) \,, \tag{77}$$

where $\varepsilon = 0_+$, and P^+ is the longitudinal part of the positronium momentum (used here as typical momentum in the problem). This defines the bracket '[ ]' in eqs. (70) and (71)

$$\frac{1}{[x]} = \frac{1}{2}\left(\frac{1}{x + i\frac{\varepsilon}{\delta_1}} + \frac{1}{x - i\frac{\varepsilon}{\delta_1}} \right) \,. \tag{78}$$

Making use of both regularizations for transversal and longitudinal components, we have for the first diagram, eq. (70),

$$\begin{aligned}\delta m^2_{1\lambda} &= p^+\delta p^-|_{p^2=m^2}\\ \delta m^2_{1\lambda} &= -\frac{e^2}{8\pi^2}\left\{3m^2\ln\left(\frac{\lambda^2\delta_1+m^2}{m^2}\right)+\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2}\left(\frac{3}{2}\lambda^2\delta_1+m^2\right)-2\lambda^2\delta_1\ln\left(\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2}\frac{\delta_1}{\varepsilon}\right)\right\}\end{aligned} \tag{79}$$

Note, that the third term has the mixing UV and IR divergencies. Combining the three relevant diagrams, fig. (6), and integrating with the common regulator, one obtains for the **electron mass correction**

$$\begin{aligned}\delta m^2_\lambda &= p^+(\delta p_1+\delta p_2+\delta p_3)|_{p^2=m^2} = -\delta\Sigma_\lambda(m^2)\\ \delta m^2_\lambda &= -\frac{e^2}{8\pi^2}\left\{3m^2\ln\left(\frac{\lambda^2\delta_1+m^2}{m^2}\right)-\frac{\lambda^2\delta_1 m^2}{\lambda^2\delta_1+m^2}\right\}\ .\end{aligned} \tag{80}$$

The mass correction is IR-finite (that gives rise to IR-finite counterterms) and contains only a logarithmic UV-divergency. Namely, when $\lambda\delta_1\to\Lambda\gg m$

$$\delta m^2_\Lambda = -\frac{3e^2}{8\pi^2}m^2\ln\frac{\Lambda^2}{m^2}\ . \tag{81}$$

It is remarkable that we reproduce with the cutoff condition of eq. (74) the standard result of covariant perturbative theory calculations including its global factor 3/8. As was mentioned above, the difference in sign, as compared with the 1-loop renormalization group result, comes from scaling down from high to low energies in the method of flow equations.

The similar regularization for the intermediate state momenta in the self-energy integrals, called 'global cutoff scheme', was introduced by W. M. Zhang and A. Harindranath [5]. In our approach the UV-regularization, that defines the concrete form of the regulator R, arises naturally from the method of flow equations, namely from the unitary transformation performed, where the generator of the transformation is chosen as the commutator $\eta=[H_d,H_r]$. Note also, that the regulator R, eq. (73), in general is independent of the electron momentum p^+ (rescaled cutoff $\lambda\delta_1\longrightarrow\lambda$), and therefore is boost invariant.

For the wave function renormalization constant, eq. (60), one has

$$\left.\frac{\partial\delta p^-}{\partial p^-}\right|_{p^2=m^2} = -\frac{e^2}{8\pi^2}\int_0^1\int d\kappa_\perp^2\left[\frac{2\frac{1}{x}-2+x}{\kappa_\perp^2+f(x)}-\frac{x(1-x)2m^2}{(\kappa_\perp^2+f(x))^2}\right]_{p^2=m^2}\times(-R)\ , \tag{82}$$

that together with the regulator R, eq. (73), results

$$\begin{aligned}Z_2 &= 1-\frac{e^2}{8\pi^2}\left\{\ln\frac{\lambda^2\delta_1}{m^2}\cdot\left(\frac{3}{2}-2\ln\frac{\delta_1}{\varepsilon}\right)+\ln\frac{\delta_1}{\varepsilon}\cdot\left(2-\ln\frac{\delta_1}{\varepsilon}\right)+F\left(\ln\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2};\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2}\right)\right\}\\ F &= \ln\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2}\left(\frac{1}{2}-\ln\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2}\right)+\frac{1}{2}\frac{\lambda^2\delta_1}{\lambda^2\delta_1+m^2}-2+2\int_0^{x_{max}}dx\frac{\ln x}{x-1}\ .\end{aligned} \tag{83}$$

As $\lambda\delta_1\to\Lambda\gg m$ the function F tends to a constant

$$F|_{\Lambda\gg m}=C=-\frac{3}{2}+\frac{\pi^2}{3}\ . \tag{84}$$

Therefore, by dropping the finite part, we obtain

$$Z_2 = 1-\frac{e^2}{8\pi^2}\left\{\ln\frac{\Lambda^2}{m^2}\cdot\left(\frac{3}{2}-2\ln\frac{1}{\varepsilon}\right)+\ln\frac{1}{\varepsilon}\left(2-ln\frac{1}{\varepsilon}\right)\right\}\ , \tag{85}$$

where we have rescaled $\frac{\varepsilon}{\delta_1} \to \varepsilon$. The electron wave function renormalization constant contains logarithmic UV and IR divergencies mixed, together with pure logarothmic IR divergencies. We mention, that the value of Z_2 is not sensitive to the form of regulator applied; the same result for Z_2 was obtained with another choice of regulator [5].

We proceed with renormalization to the second order in the **photon** sector. The diagrams that contribute to the photon self energy are shown in fig. (7). The commutator $[\eta^{(1)}, H_{ee\gamma}]$, corresponding to the first diagram, gives rise to (eq. (292) in AppendixC)

$$\begin{aligned} \delta q^-_{1\lambda}\, \delta^{ij} &= \frac{1}{[q^+]} e^2 \int \frac{d^2k^\perp dk^+}{2(2\pi)^3}\, \theta(k^+)\theta(q^+ - k^+) \\ &\quad \times Tr\left[\Gamma^i(k, k-q, q)\Gamma^j(k-q, k, -q)\right] \frac{1}{q^- - k^- - (q-k)^-} \times (-R)\ , \end{aligned} \tag{86}$$

where momenta are given in fig. (7), and the regulator is

$$R_\lambda = f^2_{q,k,\lambda} = \exp\left\{-2\left(\frac{\Delta_{q,k}}{\lambda}\right)^2\right\}\ . \tag{87}$$

In full analogy with the electron self energy this also defines the regulator for the second diagram with the instantaneous interaction, see fig. (7),

$$\delta q^-_{2\lambda}\, \delta^{ij} = \frac{1}{[q^+]} e^2 \int \frac{d^2k^\perp dk^+}{2(2\pi)^3} \theta(k^+)\, Tr(\sigma^i\sigma^j) \left(\frac{1}{[q^+ - k^+]} - \frac{1}{[q^+ + k^+]}\right) \times (-R)\ . \tag{88}$$

We define the set of coordinates

$$\begin{aligned} \frac{(q-k)^+}{q^+} &= x \\ k &= ((1-x)q^+, (1-x)q^\perp + \kappa^\perp) \\ (q-k) &= (xq^+, xq^\perp - \kappa^\perp)\ , \end{aligned} \tag{89}$$

where $q = (q^+, q^\perp)$ is the external photon momentum. Then two diagrams contribute (for details see AppendixC, eq. (296)):

$$\begin{aligned} q^+\, \delta q^-_1 &= -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa^2_\perp \\ &\quad \times \left\{ \frac{q^2}{\kappa^2_\perp + f(x)} \left(2x^2 - 2x + 1\right) + \frac{2m^2}{\kappa^2_\perp + f(x)} + \left(-2 + \frac{1}{[x][1-x]}\right)\right\} \times (-R) \\ f(x) &= m^2 - x(1-x)q^2 \\ q^+\, \delta q^-_2 &= \frac{e^2}{8\pi^2} \int_0^\infty dx \int d\kappa^2_\perp \left(\frac{1}{[1-x]} - \frac{1}{1+x}\right) \times (-R) \\ &\to -\frac{e^2}{8\pi^2} \int_0^1 dx \int d\kappa^2_\perp \frac{2}{[x]} \times (-R)\ . \end{aligned} \tag{90}$$

Note, that the transformation in the second integral is done before the regularization (by regulator the R) is performed [5].

Making use of the same approximation for the regulator as in the electron sector, we obtain for the sum of intermediate (two electron) state momenta

$$\begin{aligned} &\frac{\kappa^2_\perp + m^2}{x} + \frac{\kappa^2_\perp + m^2}{1-x} \le \lambda^2 \delta_2 \\ &\delta_2 = \frac{q^+}{P^+}\ , \end{aligned} \tag{91}$$

where the photon is put on mass-shell $q^2 = 0$ and the rescaled cutoff $\lambda \to \sqrt{2}\lambda^2/P^+$ has been used. This condition means for the transversal integration

$$\begin{aligned} R_\lambda &= \theta(\kappa^{\perp 2}_{\lambda max} - \kappa^{\perp 2})\,\theta(\kappa^{\perp 2}_{\lambda max}) \\ \kappa^{\perp 2}_{\lambda max} &= x(1-x)\lambda^2\delta_2 - m^2\ , \end{aligned} \tag{92}$$

and for the longitudinal integration

$$\begin{aligned} & x_1 \le x \le x_2 \\ & x_1 = \frac{1-r}{2} \approx \frac{m^2}{\lambda^2\delta_2} \\ & x_2 = \frac{1+r}{2} \approx 1 - \frac{m^2}{\lambda^2\delta_2} \\ & r = \sqrt{1 - \frac{4m^2}{\lambda^2\delta_2}}\ , \end{aligned} \tag{93}$$

where the approximate value is used when $m \ll \lambda$. This shows that the condition of eq. (91) for two electrons with masses m removes the light-front infrared singularities from $x \to 0$ and $x \to 1$. Thus, both UV and IR divergencies are regularized by the regulator R, eq. (92).

The mass correction arising from the first diagram, eq. (90), is

$$\delta m^2_{1\lambda} = \frac{e^2}{8\pi^2}\,\frac{2}{3}\,\lambda^2\delta_2\left(1 - \frac{4m^2}{\lambda^2\delta_2}\right)^{3/2}\ . \tag{94}$$

Combining together both diagrams with the same regulator, eq. (90), we obtain

$$\delta m^2_{\lambda} = \frac{e^2}{8\pi^2}\,\left(\frac{5}{3}\lambda^2\delta_2\,r - \frac{8}{3}m^2\,r - 2m^2\,\ln\frac{1+r}{1-r}\right)\ , \tag{95}$$

where r is defined in eq. (93). The result shows that the mass correction involves the quadratic and logarithmic UV divergencies, i.e. as $\lambda\delta_2 \to \Lambda \gg m$

$$\delta m^2_{\Lambda} = \frac{e^2}{8\pi^2}\left(\frac{5}{3}\Lambda^2 - 2m^2\,\ln\frac{\Lambda^2}{m^2}\right)\ . \tag{96}$$

The wave function renormalization constant is defined through

$$\left.\frac{\partial\delta q^-}{\partial q^-}\right|_{q^2=0} = -\frac{e^2}{8\pi^2}\int_0^1 dx\int d\kappa^2_\perp\left\{\frac{2x^2-2x+1}{\kappa^2_\perp + f(x)} + \frac{2m^2x(1-x)}{(\kappa^2_\perp + f(x))^2}\right\}\Bigg|_{q^2=0} \times(-R)\ , \tag{97}$$

that, with the regulator R, eq. (92), results

$$Z_2 = 1 - \frac{e^2}{8\pi^2}\left(-\frac{2}{3}\ln\frac{1+r}{1-r} + \frac{10}{9}\,r + \frac{8}{9}\frac{m^2}{\lambda^2\delta_2}\,r\right)\ . \tag{98}$$

The photon wave function renormalization constant contains only logarithmic UV divergency, indeed as $\lambda\delta_2 \to \Lambda \gg m$

$$Z_2 = 1 - \frac{e^2}{8\pi^2}(-\frac{2}{3}\ln\frac{\Lambda^2}{m^2}) \tag{99}$$

and is free of IR divergencies (as is expected from the form of the regulator R, eq. (91)).

3.3 Modified similarity renormalization (MSR) by means of different unitary transformations

We consider in this section the general properties of the unitary transfromation in MSR scheme, i.e. unitary transformation performed in the 'energy space' aiming to bring the field theoretical Hamiltonian to the block-diagonal form in 'particle number space'.

Basing on this analyses we construct the general form of the new interactions, generated in the effective Hamiltonian to the second order in coupling by such transformation.

3.3.1 Properties of the unitary transformation in MSR

In order to illustrate the main idea of MSR consider the flow equations in 'particle number space' [9]. Again we break the Hamiltonian into 'diagonal' and 'rest' parts, and choose for the 'diagonal' part the particle number conserving part of the Hamiltonian H_c and as basis – the states with definite number of particles

$$\begin{aligned} &H = H_d + H_r \\ &H_d = H_c \\ &H_d|n>= E_n|n> \end{aligned} \tag{100}$$

where $|n>$ is the state with particle number $'n'$.

Then the flow equations of Wegner are

$$\begin{aligned} &\frac{dH(l)}{dl} = [\eta(l), H(l)] \\ &\eta = [H_d, H_r] \end{aligned} \tag{101}$$

One has in the matrix form in particle number space

$$\begin{aligned} &\frac{dH_{mn}}{dl} = [\eta, H_r]_{mn} - (E_m - E_n)^2 H_{rmn} \\ &\eta_{mn} = (E_m - E_n) H_{rmn} \end{aligned} \tag{102}$$

that can be written through the similarity function as follows

$$\begin{aligned} &\frac{dH_{mn}}{dl} = [\eta, H_r]_{mn} + \frac{du_{mn}}{dl}\frac{H_{mn}}{u_{mn}} \\ &\eta_{mn} = \frac{1}{E_m - E_n}(-\frac{du_{mn}}{dl}\frac{H_{mn}}{u_{mn}}) \end{aligned} \tag{103}$$

where the similarity function is defined in 'particle number space' as

$$\begin{aligned} &u_{mn}(l) = \exp(-\Delta_{mn}^2 l) \\ &\Delta_{mn} = E_m - E_n \end{aligned} \tag{104}$$

According to eq. (103) (due to the second term) the 'rest' part, when $m \neq n$, behaves with the flow parameter l

$$H_{rmn} = u_{mn}\tilde{H}_{rmn} \tag{105}$$

and therefore is completly eliminated as l tends to infinity ($l \to \infty$, i.e. the band size $\lambda \to 0$). One is left with the block-diagonal effective Hamiltonian, where the number of particles is conserved in each block, fig. (3).

The obvious condition, following from eq. (104) for the similarity function in 'diagonal' sector ($m \neq n$)

$$u_d = 1 \tag{106}$$

is used further. In MSR scheme this means that the similarity function is equal to unity in particle number conserving sectors. This condition is equivalent to the choice of the particle number conserving part of Hamiltonian for the 'diagonal' part.

$$H_d = H_c \leftrightarrow u_c = 1 \tag{107}$$

The idea, that stands behind this excercise, is the similarity transformation of Glazek, Wilson but in 'particle number space', which enables to decouple many-body and few-body states in the block-diagonal effective Hamiltonian. In the 'diagonal' sectors of the effective Hamiltonian the effects of many-body states are simulated on the few-body states by a set of new interactions, induced by the similarity transformation and corresponding to the marginal relevant operators of the theory.

The problem, that arise now, is how to simulate also in the effective Hamiltonian the effects of high-energy states on the low-energy states. MSR tries to take into account both effects of high-energy and many-body states.

The idea of MSR is again to perform the similarity transformation in the 'particle number space' to bring Hamiltonian to a block-diagonal form, with the number of particles conserving in each block. Technically, we use the flow equations in the 'energy space', organized in a way to eliminate the particle number changing sectors and to generate efective Hamiltonian in the particle number conserving sectors. The effects of high-energy and many-body states are simulated then by a set of effective interactions, which do not change particle number and do not couple to high-energy states either.

The 'diagonal' sector is the particle number conserving part of the Hamiltonian, and (or) the similarity function in the particle number conserving sector is equal to unity

$$H_d = H_c \leftrightarrow u_{dij} = 1 \tag{108}$$

The flow equations are written in the basis of single particle states $|i>$, i.e.

$$H_0|i>= E_i|i> \tag{109}$$

where H_0 is free (noninteracting) part of Hamiltonian.

We write the flow equations explicitly in the perturbative frame, i.e. break the Hamiltonian

$$H = H_{0d} + \sum_n (H_d^{(n)} + H_r^{(n)}) \tag{110}$$

where $H^{(n)} \sim e^n$ and the indices 'd' ('r') correspond to 'diagonal' ('rest') sectors (here we do not refer to the definite field theory). The flow equation and generator of transformation to the 'n'-th order are

$$\begin{aligned} \frac{dH^{(n)}}{dl} &= \sum_k [\eta^{(k)}, H_d^{(n-k)} + H_r^{(n-k)}] + \sum_k [[H_d^{(k)}, H_r^{(n-k)}], H_{0d}] + [[H_{0d}, H_r^{(n)}], H_{0d}] \\ \eta^{(n)} &= [H_{0d}, H_r^{(n)}] + \sum_k [H_d^{(k)}, H_r^{(n-k)}] \end{aligned} \tag{111}$$

This is written in the matrix form in the basis of states $|i>$ as

$$\frac{dH_{ij}^{(n)}}{dl} = \sum_k [\eta^{(k)}, H_d^{(n-k)} + H_r^{(n-k)}]_{ij} \\ -(E_i - E_j)\sum_k [H_d^{(k)}, H_r^{(n-k)}]_{ij} - (E_i - E_j)^2 H_{rij}^{(n)} \qquad (112)$$

In 'diagonal' and 'rest' sectors one has corresponding

'diagonal sector'

$$\frac{dH_{dij}^{(n)}}{dl} = \sum_k [\eta^{(k)}, H_d^{(n-k)} + H_r^{(n-k)}]_{dij} + \sum_k [[H_d^{(k)}, H_r^{(n-k)}]_d, H_{0d}]_{ij} \qquad (113)$$

'rest sector'

$$\frac{dH_{rij}^{(n)}}{dl} = \sum_k [\eta^{(k)}, H_d^{(n-k)} + H_r^{(n-k)}]_{rij} \\ + \sum_k [[H_d^{(k)}, H_r^{(n-k)}]_r, H_{0d}]_{ij} - (E_i - E_j)^2 H_{rij}^{(n)} \qquad (114)$$

The main difference between these two sectors is the presence of the third term in the 'rest' sector

$$-(E_i - E_j)^2 H_{rij}^{(n)} = \frac{du_{rij}}{dl} \cdot \frac{H_{rij}^{(n)}}{u_{rij}} \qquad (115)$$

that gives rise to the band-diagonal structure in the 'energy space' for the 'rest' part of effective Hamiltonian, i.e.

$$H_{rij} = u_{rij}\tilde{H}_{rij} \\ u_{rij} = \mathrm{e}^{-(E_i - E_j)^2 l} \qquad (116)$$

When $l \to \infty$ the 'rest' part is completly eliminated, except for the diagonal in 'energy space' matrix elements $i = j$, which do not contribute to physical values (see further). One assumes therefore, that different Fock states are decoupled in the block-diagonal effective Hamiltonian.

In the 'diagonal' sector the matrix elements with any energy differences are present, $u_{dij} = 1$. According to eq. (113) the new terms are induced in this sector, containing in the lowest Fock sectors both the information on high-Fock components (included due to the presence in the canonical Hamiltonian the interactions that mix different Fock components) and high-energy states.

The renormalized effective Hamiltonian contains the nonlocal new interactions, corresponding to marginal relevant operators, and local operators – counterterms, defined order by order in coupling constant. One is able therefore to perform at a time renormalization programm and work in the lowest Fock sectrors to solve bound state problem.

The property of the unitary transformation in MSR scheme, we are going to use in the next section, is the difference between the 'diagonal' and 'rest' sectors, given by the following expressions for similarity functions in both sectors

$$u_{dij} = 1 \\ u_{rij} \neq 1 \qquad (117)$$

3.3.2 Generated interaction in MSR

Here we calculate the interactions in light-front QED_{3+1}, generated to the second order in coupling by different unitary transformations of MSR scheme. Flow equations and similarity transformations of Glazek, Wilson, acting in SR scheme, are summarized in Appendix A.

The second order interaction, generated by flow equations in MSR in the 'diagonal' sector, is

$$\frac{dH^{(2)}_{dij}}{dl} = [\eta^{(1)}, H^{(1)}_r]_{dij}$$
$$\eta^{(1)}_{ij} = -\frac{1}{E_i - E_j}\frac{dH^{(1)}_{rij}}{dl} \tag{118}$$

where the indices 'd' and 'r' define 'diagonal' (Fock state conserving) and 'rest' (Fock state changing) sectors corresponding. The commutator can be written

$$[\eta^{(1)}, H^{(1)}_r]_{dij} = -\sum_k \left(\frac{\frac{dH^{(1)}_{rik}}{dl} H^{(1)}_{rjk}}{E_i - E_k} + \frac{\frac{dH^{(1)}_{rjk}}{dl} H^{(1)}_{rik}}{E_j - E_k} \right) \tag{119}$$

Neglecting the dependence of energy on the flow parameter, one has the following generated interaction

$$H^{(2)}_{dij}(\lambda) = H^{(2)}_{dij}(\lambda = \Lambda \to \infty) + \sum_k \left(\frac{\int_\lambda^\infty \frac{dH^{(1)}_{rik}}{d\lambda'} H^{(1)}_{rjk} d\lambda'}{E_i - E_k} + \frac{\int_\lambda^\infty \frac{dH^{(1)}_{rjk}}{d\lambda'} H^{(1)}_{rik} d\lambda'}{E_j - E_k} \right) \tag{120}$$

where the connection $l = 1/\lambda^2$ is used. Making use of the first order solution through similarity function

$$H^{(1)}_{rij}(\lambda) = H^{(1)}_{rij}(\lambda = \Lambda \to \infty) \frac{f_{ij}(\lambda)}{f_{ij}(\lambda = \Lambda \to \infty)} \tag{121}$$

one has

$$H^{(2)}_{dij}(\lambda) = H^{(2)}_{dij}(\lambda = \Lambda \to \infty) + \sum_k H^{(1)}_{rik}(\lambda = \Lambda \to \infty) H^{(1)}_{rjk}(\lambda = \Lambda \to \infty) \cdot$$
$$\left(\frac{\int_\lambda^\infty \frac{df^{(1)}_{ik}(\lambda')}{d\lambda'} f^{(1)}_{jk}(\lambda') d\lambda'}{E_i - E_k} + \frac{\int_\lambda^\infty \frac{df^{(1)}_{jk}(\lambda')}{d\lambda'} f^{(1)}_{ik}(\lambda') d\lambda'}{E_j - E_k} \right) \tag{122}$$

Note, that first and second order flow equations eq. (118) in 'diagonal' sector, i.e. when the condition $u_{dij} = 1$ is implied, coincide for both MSR eqs. (112) and (113) and SR eq. (234)a4 of AppendixA schemes. Assuming $u_{dij} = 1$ for other similarity transformations of Appendix A, we obtain the same form of interaction, namely eq. (122), generated to the second order in 'diagonal' sector by any unitary transformation in MSR scheme.

We proceed along the same line in the light-front frame. Basing on calculations of section 3.2.1, we get the following generated interaction in MSR scheme (for exchange and annihilation channels)

$$V^{(ex)}_{gen}(\lambda) = -e^2 M^{(ex)}_{2ii} \frac{1}{(p^+_1 - p^+_3)} \left(\frac{\int_\lambda^\infty \frac{df_{p_1,p_3,\lambda'}}{d\lambda'} f_{p_4,p_2,\lambda'} d\lambda'}{\Delta_{p_1,p_3}} + \frac{\int_\lambda^\infty \frac{df_{p_4,p_2,\lambda'}}{d\lambda'} f_{p_1,p_3,\lambda'} d\lambda'}{\Delta_{p_4,p_2}} \right) \tag{123}$$

$$V^{(an)}_{gen}(\lambda) = e^2 M^{(an)}_{2ii} \frac{1}{(p^+_1 + p^+_2)} \left(\frac{\int_\lambda^\infty \frac{df_{p_1,-p_2,\lambda'}}{d\lambda'} f_{p_4,-p_3,\lambda'} d\lambda'}{\Delta_{p_1,-p_2}} + \frac{\int_\lambda^\infty \frac{df_{p_4,-p_3,\lambda'}}{d\lambda'} f_{p_1,-p_2,\lambda'} d\lambda'}{\Delta_{p_4,-p_3}} \right)$$

One obtaines the explicit form of generated interaction, induced by one of the unitary transformations in MSR scheme, specifying the similarity function f_{ij} in eq. (124) (Apendix A). Namely, one has for the flow equations of Wegner [1]

$$f_{p_1p_2} = u_{p_1p_2} = \mathrm{e}^{-\frac{\Delta^2_{p_1p_2}}{\lambda^2}} \tag{124}$$

and for the similarity transformation of Glazek,Wilson two alternative forms [2],[17]

$$\begin{aligned} f_{p_1p_2} &= u_{p_1p_2}\mathrm{e}^{r_{p_1p_2}} \\ f_{p_1p_2} &= u_{p_1p_2} \end{aligned} \tag{125}$$

where

$$u_{p_1p_2} = \theta(\lambda - |\Delta_{p_1p_2}|) \tag{126}$$

We use the general form of generated interaction eq. (124) in the light front frame for the further analyses in section 4.3.

3.4 Renormalized to the second order effective QED_{3+1} Hamiltonian on the light front

In this section we summarize the results obtained in the previous sections. Basing on the flow equations we have completed the renormalization of light front QED (LFQED) together with calculation of effective Hamiltonian to the second order in coupling constant.

Consider first the situation when the 'rest' sector is not completly eliminated, i.e. canonical electron-photon coupling (the matrix elements) is still present in the energy band $|E_i - E_j| < \lambda$. This means that perturbative photons are present in the effective Hamitonian and one is able therefore to formulate the diagramatic rules for the perturbative expansion in coupling.

The matrix elements of the renormalized to the second order effective Hamitonian, namely the diagrams in different Fock sectors, are depicted in Table 1, and corresponding analytical expressions for 'diagonal' and 'rest' sectors are listed in fig. (5). The diagramatic rules are obtained by direct calculation of matrix elements between free particle states.

Instantaneous diagrams as light front gauge artefact are not included in MSR procedure [13],[14]. This means that instantaneous terms (also belonging to the 'rest' sectors) stay intact with the unitary transformation (i.e. do not obtain the similarity factor).

The matrix elements of the 'rest' (Fock state changing) sectors are sqeezed in the energy band $\Delta_{p_ip_f} = |\sum p_i^- - \sum p_f^-| < \lambda$; the 'diagonal' (Fock state conserving) sectors, exist for any energy differences. In the Table 1 we denote as dot the zero to the second order matrix elements.

Using flow equations one has to the second order in coupling the following matrix elements in the **'diagonal' sectors** between the Fock states

$$\begin{aligned} &|e\bar{e}> \rightarrow |e\bar{e}>, |e\bar{e}e\bar{e}> \rightarrow |e\bar{e}e\bar{e}>, ... \\ &-e^2_\lambda M_{2ij,\lambda}\delta^{ij}\frac{1}{[p_1^+ - p_3^+]}\left(\frac{\int_\lambda^\infty \frac{df_{p_1p_3\lambda'}}{d\lambda'}f_{p_4p_2\lambda'}d\lambda'}{\Delta_{p_1p_3\lambda}} + \frac{\int_\lambda^\infty \frac{df_{p_4p_2\lambda'}}{d\lambda'}f_{p_1p_3\lambda'}d\lambda'}{\Delta_{p_4p_2\lambda}}\right) \\ &|e\bar{e}\gamma> \rightarrow |e\bar{e}\gamma>, |e\bar{e}\gamma\gamma> \rightarrow |e\bar{e}\gamma\gamma>, ... \end{aligned}$$

$$e^2_\lambda \tilde{M}_{2ij,\lambda}\varepsilon^{i*}\varepsilon^j\left(\frac{\int_\lambda^\infty \frac{df_{p_1k_1\lambda'}}{d\lambda'}f_{p_2k_2\lambda'}d\lambda'}{\Delta_{p_1k_1\lambda}} + \frac{\int_\lambda^\infty \frac{df_{p_2k_2\lambda'}}{d\lambda'}f_{p_1k_1\lambda'}d\lambda'}{\Delta_{p_2k_2\lambda}}\right), \tag{127}$$

and in the **'rest' sectors** between the Fock states

$$|e\bar{e}>\rightarrow |e\bar{e}e\bar{e}>, |e\bar{e}e\bar{e}>\rightarrow |e\bar{e}>, ...$$
$$-e_\lambda^2 f_{p_i p_f \lambda} M_{2ij,\lambda}\delta^{ij}\frac{1}{[p_1^+ - p_3^+]}\left(\frac{\int_\lambda^\infty \frac{1}{f_{p_i p_f \lambda'}}\frac{df_{p_1 p_3 \lambda'}}{d\lambda'} f_{p_4 p_2 \lambda'} d\lambda'}{\Delta_{p_1 p_3 \lambda}} + \frac{\int_\lambda^\infty \frac{1}{f_{p_i p_f \lambda'}}\frac{df_{p_4 p_2 \lambda'}}{d\lambda'} f_{p_1 p_3 \lambda'} d\lambda'}{\Delta_{p_4 p_2 \lambda}}\right)$$

$$|e\bar{e}>\rightarrow |\gamma\gamma>, |\gamma\gamma>\rightarrow |e\bar{e}>, ...$$
$$e_\lambda^2 f_{p_i p_f \lambda}\tilde{M}_{2ij,\lambda}\varepsilon^{i*}\varepsilon^j\left(\frac{\int_\lambda^\infty \frac{1}{f_{p_i p_f \lambda'}}\frac{df_{p_1 k_1 \lambda'}}{d\lambda'} f_{p_2 k_2 \lambda'} d\lambda'}{\Delta_{p_1 k_1 \lambda}} + \frac{\int_\lambda^\infty \frac{1}{f_{p_i p_f \lambda'}}\frac{df_{p_2 k_2 \lambda'}}{d\lambda'} f_{p_1 k_1 \lambda'} d\lambda'}{\Delta_{p_2 k_2 \lambda}}\right) , \quad (128)$$

All momenta are given in fig. (5) We plug the similarity function

$$f_{p_i p_f \lambda} = \mathrm{e}^{-\frac{\Delta_{p_i p_f}^2}{\lambda^2}}$$
$$\Delta_{p_i p_f} = \sum p_i^- - \sum p_f^- \ , \quad (129)$$

into eq. (127),eq. (128) to get the explicit form of interactions in both sectors listed in fig. (5). 'Rest' diagrams are drawn schematically, to show the difference between the interactions in 'diagonal' and 'rest' sectors. Namely, for the 'rest' sectors we imply, that the corresponding momentum exchange must be done in the diagrams of fig. (5) to get analytical expressions for diagrams depicted in Table 1.

In the diagramatic rules we write explicitly the dependence of the electron (photon) mass on the cutoff

$$m_\lambda^2 = m_0^2 - \delta\Sigma_\lambda \ , \quad (130)$$

where $\delta\Sigma_\lambda$ is the self energy term to the second order ($m_0 = 0$ for a photon); we drop the subscript λ for the polarization vectors ε and spinors χ. To the next, third, order one has $e_\lambda = e_0(1 + O(e_\lambda^2))$.

The sense of the Table 1 is transparent. Elimination to the second order of canonical electron-photon vertex with flow equations gives rise to one-particle and two-particle operators. The elimination of the 'rest' sectors to the next orders generate many-particle operators. One is able therefore to trancate the effective Hamitonian to the few-particle sectors only in the case when the perturbative theory expansion is true.

As $\lambda \rightarrow 0$ the 'rest' sectors are completly (here to the order $O(e^2)$) eliminated and Fock states in 'diagonal' sectors are decoupled. One ends up with the renormalized effective LFQED Hamiltonian.

The two-component LF theory, introduced by Zhang and Harindranath [5], as compared with four-component formalism of Brodsky and Lepage, is formulated purely in terms of physical degrees of freedom; so that each term in the renormalized effective Hamiltonian corresponds to a real dynamical process (or give renormalization term). Therefore the 'diagonal' sectors of the renormalized effective Hamiltonian contribute for different Fock states to: $|\gamma>$ - self energy photon operator, $|e\bar{e}>$ - electron-positron bound state (or scattering), $|\gamma\gamma>$ - light-light scattering, $|e\bar{e}\gamma>$ - Compton scattering, and so on (see Table 1).

In the next section we use the diagramatic rules given in the Table 1 (at $\lambda \neq 0$) to calculate the effective electron-positron interaction, which includes generated interaction, instantaneous term and perturbative photon exchage.

4 Positronium bound state

4.1 Light front perturbative theory

The scattering $|e\bar{e}>$ states are also needed in bound state calculations. Using the propagator techniques we include these states where required. We exploit the perturbative theory in the coupling constant e, using the diagramatic rules for the renormalized effective theory fig. (5).

The first order renormalized $ee\gamma$-vertex $f_{p_ip_f}H^{ee\gamma}_{can}$ contributes to the second order to the $|e\bar{e}>$ interaction term and to the electron (photon) mass renormalization. Physically, it is the perturbative photon exchange (photon emission and absorbtion in the case of electron mass renormalization), with the energy widths of the photon restricted by the function $f_{p_ip_f,\lambda}$.

4.1.1 Electron-positron interaction

According to the light-front Feynman rules the perturbative photon exchange gives rise to the following second order $|e\bar{e}>$ interaction in the exchange channel

$$V(l) = g_1(l)g_2(l)M_{2ii}\cdot\left\{\frac{\theta(q^+)}{q^+}\frac{1}{p_i^- - p_k^-} + \frac{\theta(-q^+)}{(-q^+)}\frac{1}{p_i^- - p_k^-}\right\}\ , \tag{131}$$

where g_i stands schematically for the coupling constants in both vertices, namely $g_{p_1p_2\lambda} = ef_{p_1p_2\lambda}$ and M_{2ii} defines the spin structure of the interaction, coming from the corresponding structure of the $ee\gamma$-vertex; and the two terms in the curly brackets represent two different x^+ (time) orderings of the photon exchange with the momentum q, giving rise to the two different intermediate states with momenta p_k, p_i corresponds to the initial state. Explicitly one has

In the **exchange channel**

$$\begin{aligned} V^{(ex)}_{PT}(l) &= -e^2(f_{-p_4,-p_2}(l)\chi^+_{\bar{s}_2}\Gamma^i(-p_4,-p_2,-q)\chi_{\bar{s}_4})(f_{p_1,p_3}(l)\chi^+_{s_3}\Gamma^i(p_1,p_3,q)\chi_{s_1}) \\ &\times\left[\frac{\theta(p_1^+-p_3^+)}{(p_1^+-p_3^+)}\frac{1}{p_i^- - p_3^- - p_2^- - q^-} + \frac{\theta(p_3^+-p_1^+)}{(p_3^+-p_1^+)}\frac{1}{p_i^- - p_1^- - p_4^- + q^-}\right]\bar{\delta}_{q,p_1-p_3} \end{aligned} \tag{132}$$

with the initial state momentum $p_i = P = (P^+, P^\perp)$ and momentum transfer $q = p_1 - p_3$, and

$$P^- = \frac{P^{\perp 2} + M_N^2}{P^+}\ , \tag{133}$$

where M_N is the mass of positronium bound state. In the light-front frame holds

$$\begin{aligned} -(P^- - p_3^- - p_2^- - (p_1-p_3)^-)\theta(p_1^+ - p_3^+) &= (P^- - p_1^- - p_4^- + (p_1-p_3)^-)\theta(p_3^+ - p_1^+) \\ &= \frac{\tilde{\Delta}_3}{P^+(x-x')} \end{aligned} \tag{134}$$

that gives rise for the rescaled potential $V = \tilde{V}/P^{+2}$

$$\tilde{V}^{(ex)}_{PT,\lambda} = -e^2N_1\frac{1}{\tilde{\Delta}_3}\exp\left(-\frac{(\Delta_1^2+\Delta_2^2)}{\lambda^4}\right)\ , \tag{135}$$

where N_1 is defined in eq. (55) and

$$\begin{aligned} \tilde{\Delta}_3 &= (k_\perp - k'_\perp)^2 + \frac{1}{2}(x-x')A + |x-x'|\left(\frac{1}{2}(M_0^2 + M_0'^2) - M_N^2\right) \\ A &= (k_\perp^2+m^2)\left(\frac{1}{1-x} - \frac{1}{x}\right) + (k_\perp'^2+m^2)\left(\frac{1}{x'} - \frac{1}{1-x'}\right)\ , \end{aligned} \tag{136}$$

Here the cutoff λ is defined in units of P^+, namely $\lambda \to \lambda^2/P^+$.

Because of the absence of Z-graphen in light-front formalism (corresponding to negative p^+), only one term contribute to the **annihilation channel**, namely

$$\begin{aligned} V_{PT}^{(an)}(l) &= e^2(f_{p_1,-p_2}(l)\chi^+_{\bar{s}_2}\Gamma^i(p_1,-p_2,q)\chi_{s_1})(f_{-p_4,p_3}(l)\chi^+_{s_3}\Gamma^i(-p_4,p_3,-q)\chi_{\bar{s}_4}) \\ &\times\left[\frac{1}{(p_1^+ + p_2^+)}\frac{1}{p_i^- - q^-}\right]\bar{\delta}_{q,p_1+p_2}\ , \end{aligned} \tag{137}$$

where $p_i^- = P^-$ and the momentum transfer is $q = p_1 + p_2$. This gives rise for the rescaled potential $V = \tilde{V}/P^{+2}$ in the light-front frame, to the expression

$$\tilde{V}^{(an)}_{PT,\lambda} = e^2 N_2 \frac{1}{M_N^2}\exp\left\{-\frac{(M_0^4 + M_0'^4)}{\lambda^4}\right\}\ , \tag{138}$$

where N_2 and the variables Δ_1, Δ_2 and M_0^2, M'^2_0 are defined in eq. (55).

4.1.2 Mass renormalization

Following light-cone rules the perturbative energy correction of the electron with momentum p, coming from the emission and absorption of a photon with momentum k, is

$$\begin{aligned} \delta\tilde{p}^-_{1\lambda} = \int \frac{d^2k^\perp dk^+}{2(2\pi)^3}\frac{\theta(k^+)}{k^+}\theta(p^+ - k^+) g_{p-k,p,\lambda}\Gamma^i_\lambda(p-k,p,-k) g_{p,p-k,\lambda}\Gamma^i_\lambda(p,p-k,k) \\ \times\frac{1}{p^- - k^- - (p-k)^-}\ , \end{aligned} \tag{139}$$

where $g_{ee\gamma}$-coupling constant restricts the energy of the photon. Making use of the explicit form for the coupling, one has

$$\begin{aligned} \delta\tilde{p}^-_{1\lambda} &= e^2\int \frac{d^2k^\perp dk^+}{2(2\pi)^3}\frac{\theta(k^+)}{k^+}\theta(p^+ - k^+) \\ &\times\Gamma^i_\lambda(p-k,p,-k)\,\Gamma^i_\lambda(p,p-k,k)\frac{1}{p^- - k^- - (p-k)^-}\times(R)\ , \end{aligned} \tag{140}$$

where $R = f^2_{pk\lambda}$ plays the role of regulator. This expression coincide up to the overall sign with the energy correction obtained in the previous section from the flow equations method.

Two instantaneous diagramms, arising from the normal-ordering Hamiltonian, must be added to the first term with the same regulator R. Then the full perturbative energy correction $\delta\tilde{p}^-_\lambda = \delta\tilde{p}^-_{1\lambda} + \delta\tilde{p}^-_{2\lambda} + \delta\tilde{p}^-_{3\lambda}$ is

$$\delta\tilde{p}^-_\lambda = -\delta p^-_\lambda \tag{141}$$

where δp^-_λ is defined in eq. (63). This means for the perturbative mass correction

$$\delta m^{PT2}_\lambda = \delta\Sigma_\lambda \tag{142}$$

and the self-energy term $\delta\Sigma_\lambda$ is given in eq. (80).

We combine the renormalized to the second order mass, eq. (62), and the perturbative correction, eq. (142), to obtain the total physical mass to the order $O(e^2)$

$$m_e^2 = m^2_\lambda + \delta m^2 = (m^2 + \delta\Sigma_\lambda) - \delta\Sigma_\lambda = m^2 + O(e^4)\ . \tag{143}$$

This means, that to the second order $O(e^2)$ the physical electron mass is, up to a finite part, equal to the bare electron mass, that stands in the free (canonical) Hamiltonian.

Along the same line one can do for the photon mass.

At the end we note, that the similarity function $f_{p_i p_f \lambda}$, restricting the electron-photon vertex, plays the role of UV (and partialy IR) regulator in the self energy integrals. This means, that the regularization prescription of divergent integrals follows from the method of flow equations itself. Moreover, the energy correction (i.e. mass correction and wave function renormalization constant), obtained from the flow equations, coincide up to the overall sign with the 1-loop renormalization group result. This is the remarkable result, indicating to the equivalence of flow equations and Wilson's renormalization.

4.2 Bound state perturbative theory

In this subsection we define bound state perturbative theory (BSPT).

First introduce instead of the light front parametrization, used before for the single-particle momenta, fig. (8), the instant form

$$\begin{gathered} p_{1\mu} = (xP^+, xP^\perp + k^\perp, p_1^-) \quad \stackrel{J(p)}{\longrightarrow} \quad p_{1\mu} = (k_z, k^\perp, p_1^0) = (\vec{p}_1, E_1) \\ p_{2\mu} = ((1-x)P^+, (1-x)P^\perp - k^\perp, p_2^-) \quad \longrightarrow \quad p_{2\mu} = (-k_z, -k^\perp, p_2^0) = (\vec{p}_2, E_2) \\ E_i = \sqrt{\vec{p}^2 + m^2}, \quad i = 1, 2 \\ x = \frac{E_1 + k_z}{E_1 + E_2} = \frac{1}{2}\left(1 + \frac{k_z}{\sqrt{\vec{p}^2 + m^2}}\right) \end{gathered} \tag{144}$$

and for the momenta p_3, p_4 the same, but with prime over $x, k_z, k^\perp$; here x is the light-front fraction of the electron momentum, and $J(p)$ is the Jacobian of the transformation:

$$J(p) = \frac{dx}{dk_z} = \frac{k_\perp^2 + m^2}{2(\vec{p}^2 + m^2)^{3/2}} \; . \tag{145}$$

The instant form system is usefull from the practical point of view: it is easy to see then the rotational symmetry, restored in the nonrelativistic case and manifest in positronium spectrum.

Define now **BSPT**. We choose the leading order electron-positron potential in such a form to simplify positronium bound state calculations. This means, that this potential contributes the leading order term to the positronium mass, and perturbative theory with respect to the difference between the total second order $|e\bar{e}>$ interaction, calculated before with the renormalized Hamiltonian to $O(e^2)$, and the leading order potential converges. This scheme we call BSPT. Surely, our choice is motivated by the form of the renormalized to the second order interaction to insure convergence of BSPT.

In MSR scheme, where the 'diagonal' part is given by the Fock state conserving part of the Hamiltonian, the choice of the zero'th order potential for BSPT calculations is quit natural. It corresponds to the leading order in coupling of effective Hamiltonian in 'diagonal' sector. This insures the convergence of bound state calculations. Note, that in SR scheme the 'diagonal' part is given by free Hamiltonian that can not surve for the starting approximation in bound state perturbative theory.

We define the second order renormalized electron-positron potential $\langle e(3)\bar{e}(4)| \hat{V}_{coul} |e(1)\bar{e}(2)\rangle$ to the leading order of BSPT in the form of pure perturbative one photon exchange, explicitly as the Coulomb interaction

$$V_{coul} = -\frac{16e^2m^2}{(k_\perp - k'_\perp)^2 + (k_z - k'_z)^2} = -\frac{16e^2m^2}{(\vec{p} - \vec{p'})^2} \; . \tag{146}$$

This means that the corresponding leading order Hamilton operator in the $|e\bar{e}>$ sector is

$$H^{(0)} = h + \hat{V}_{coul} \ , \tag{147}$$

where h is the free part, defined in eq. (29). The wave functions are given as the solution of Schrödinger equation

$$H^{(0)}|\psi_N(P)>= E_N|\psi_N(P)> \ , \tag{148}$$

where P is the positronium momentum, and the eigenvalues and eigenfunctions for the positronium bound state are defined in standard way of light front frame

$$\begin{aligned} E_N &= \tfrac{P_\perp^2+M_N^2}{P^+} \\ |\psi_N(P)>= \textstyle\sum_{s_1s_2} \int_{p_1p_2} \sqrt{p_1^+p_2^+}\, 2(2\pi)^3\, \delta^{(3)}(P-p_1p_2)\, \tilde{\Phi}_N(xk_\perp s_1s_2)\, b^+_{s_1}(p_1)\, d^+_{s_2}(p_2)|0> \\ \textstyle\sum_{s_1s_2} \frac{\int d^2k_\perp \int_0^1 dx}{2(2\pi)^3}\, \tilde{\Phi}^*_N(xk_\perp s_1s_2)\, \tilde{\Phi}'_N(xk_\perp s_1s_2) = \delta_{NN'} \end{aligned} \tag{149}$$

M_N stands for the leading order mass of positronium. Combining the definitions for the wave function and the energy with the Schrödinger equation, we obtain

$$\left[M_N^2 - \frac{k_\perp'^2+m^2}{x'(1-x')}\right]\tilde{\Phi}_N(x'k'_\perp s_3s_4) = \sum_{s_1s_2} \frac{\int d^2k_\perp \int_0^1 dx}{2(2\pi)^3}\, V_{coul}\, \tilde{\Phi}_N(xk_\perp s_1s_2) \ , \tag{150}$$

or, after change of coordinates according to eq. (144),

$$\left(M_N^2 - 4(\vec{p'}^{\,2}+m^2)\right)\Phi_N(\vec{p'}s_3s_4) = \sum_{s_1s_2} \frac{\int d^3p \sqrt{J(p)J(p')}}{2(2\pi)^3}\, V_{coul}(\vec{p},\vec{p'})\, \Phi_N(\vec{p}s_1s_2) \ , \tag{151}$$

where the wave function was redefined to have the norm

$$\sum_{s_1s_2} \int d^3p\, \Phi^*_N(\vec{p}s_1s_2)\, \Phi'_N(\vec{p}s_1s_2) = \delta_{NN'} \ . \tag{152}$$

We aim to obtain the nonrelativistic Schrödinger equation for positronium. Note, that in the nonrelativistic limit $\frac{\vec{p}^2}{m^2} \ll 1$ we have

$$\begin{aligned} \sqrt{J(p)J(p')} \ &\approx \ \tfrac{1}{2m}\left(1 - \tfrac{\vec{p}^2+(k_z^2+k_z'^{\,2})}{2m^2}\right) \\ M_N \ &= \ (2m+B_N)^2 \ \approx \ 4m^2 + 4mB_N^{(0)} \quad , \end{aligned} \tag{153}$$

where we have introduced the leading order binding energy $B_N^{(0)}$. Then to the leading order the bound state equation for positronium is

$$\left(\frac{\vec{p'}^{\,2}}{m} - B_N\right)\Phi_N(\vec{p'}s_3s_4) \ = \ -\sum_{s_1s_2}\int d^3p \left(\frac{1}{2m}\frac{1}{2(2\pi)^3}\frac{1}{4m}\, V_{coul}\right)\, \Phi_N(\vec{p}s_1s_2) \ . \tag{154}$$

Making use of the explicit form for the Coulomb potential, eq. (146), we obtain the equation that determines the leading order bound state wave function:

$$\left(\frac{\vec{p'}^{\,2}}{m} - B_N\right)\, \Phi_\mu(\vec{p'}) \ = \ \frac{\alpha}{2\pi^2}\int \frac{d^3p}{(\vec{p}-\vec{p'})^2}\, \Phi_\mu(\vec{p}) \tag{155}$$

with

$$\Phi_N \;=\; \Phi_{\mu,s_e,s_{\bar{e}}}(\vec{p'}s_3s_4) \;=\; \Phi_\mu(\vec{p'})\,\delta_{s_es_3}\,\delta_{s_{\bar{e}}s_4}\;. \tag{156}$$

This is the standard nonrelativistic Schrödinger equation for positronium. Its solution is characterized by $\mu=(n,l,m)$, the usual principal and angular momentum quantum numbers. The wave functions are given through the hyperspherical harmonics

$$\begin{aligned}
Y_\mu(\Omega) \;&=\; \frac{(e_n^2+\vec{p}^2)^2}{4\,e_n^{5/2}}\,\Phi_\mu \\
Y_\mu \;&=\; Y_{n,l,m} \;=\; f_{n,l}(\omega)\,Y_{l,m}(\theta,\phi) \\
B_N \;&=\; -\frac{m\alpha^2}{4n^2}, \qquad e_n \;=\; \frac{m\alpha}{2n}
\end{aligned} \tag{157}$$

and for the binding energy one has the standard nonrelativistic expression for positronium bound state to $O(e^2)$. For sake of completeness we write the coordinates used in the solution

$$\begin{aligned}
&(e_n^2=-mB_N,\vec{p}) \;\longrightarrow\; (u_0,\vec{u}) \\
&u_0 \;=\; \cos\omega \;=\; \frac{e_n^2-\vec{p}^2}{e_n^2+\vec{p}^2} \\
&\vec{u} \;=\; \frac{\vec{p}}{|\vec{p}|}\sin\omega \;=\; \frac{2e_n\vec{p}}{e_n^2+\vec{p}^2}\;,
\end{aligned} \tag{158}$$

but, for details, refer to [6].

The electron-positron interaction arising from the renormalized to the $O(e^2)$ Hamiltonian is given as a sum of two contributions from exchange and annihilation channels $V=V_{exch}+V_{ann}$ (see explicitly later). We introduce the potential, arising in the nonrelativistic Schrödinger equation, eq. (155),

$$V'(\vec{p'}s_3s_4;\vec{p}s_1s_2) = \lim_{\frac{\vec{p}^2}{m^2}<<1} \frac{\sqrt{J(p)J(p')}}{2(2\pi)^3}\frac{1}{4m}(\tilde{V}_{exch}+\tilde{V}_{ann})\;. \tag{159}$$

Then we define BSPT with respect to the difference

$$\delta V = V'(\vec{p'}s_3s_4;\vec{p}s_1s_2) - (-\frac{\alpha}{2\pi^2})\frac{1}{(\vec{p}-\vec{p'})^2}\delta_{s_1s_3}\delta_{s_2s_4}\;, \tag{160}$$

where the leading order contribution is defined in eq. (157). Note, that, in order to define the Coulomb potential, i.e. the $e\bar{e}$ interaction to the leading order of BSPT, we have taken only the first term of nonrelativistic expansion of the Jacobian $J(p)$.

In what follows we use the matrix elements of δV, defined as

$$<\Phi_{nlm}|\delta V|\Phi_{nlm}>=\int d^3pd^3p'\Phi^*_{nlm}(\vec{p})\delta V\Phi_{nlm}(\vec{p'})\;, \tag{161}$$

where Φ_{nlm} are the Coulomb wave functions given above.

4.3 Renormalized effective electron-positron interaction

In this section we consider the properties of effective electron-positron interaction, obtained by flow equations and also by the general unitary transfromation in MSR scheme.

4.3.1 Renormalized effective electron-positron interaction in light front and instant frames

We summarize together all the terms defining the electron-positron interaction, obtained by flow equations, in exchange and annihilation channels.

$$\begin{aligned}\tilde{V}_{exch} &= \tilde{V}_\lambda^{exch} + \tilde{V}^{PT} = \tilde{V}_\lambda^{gen} + \tilde{V}_\lambda^{inst} + \tilde{V}_\lambda^{PT} \\ \tilde{V}_{ann} &= \tilde{V}_\lambda^{ann} + \tilde{V}^{PT} = \tilde{V}_\lambda^{gen} + \tilde{V}_\lambda^{inst} + \tilde{V}_\lambda^{PT} \ , \end{aligned} \tag{162}$$

First we use the **light front frame**. The generated, instantaneous and perturbative theory interactions (rescaled, i.e. $V_\lambda = \tilde{V}_\lambda / P^{+2}$) are given corresponding in the **exchange channel**

$$\begin{aligned}\tilde{V}_\lambda^{gen} &= -e^2 N_{1,\lambda} \left(\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\tilde{\Delta}_1^2 + \tilde{\Delta}_2^2} \right) \left(1 - \mathrm{e}^{-\frac{\Delta_1^2 + \Delta_2^2}{\lambda^4}} \right) \\ \tilde{V}_\lambda^{inst} &= -\frac{4e^2}{(x-x')^2} \, \delta_{s_1 s_3} \, \delta_{s_2 s_4} \\ \tilde{V}_\lambda^{PT} &= -e^2 N_{1,\lambda} \frac{1}{\tilde{\Delta}_3} \mathrm{e}^{-\frac{\Delta_1^2 + \Delta_2^2}{\lambda^4}} \end{aligned} \tag{163}$$

in the **annihilation channel**

$$\begin{aligned}\tilde{V}_\lambda^{gen} &= e^2 N_{2,\lambda} \left(\frac{M_0^2 + M_0'^2}{M_0^4 + M_0'^4} \right) \left(1 - \mathrm{e}^{-\frac{M_0^4 + M_0'^4}{\lambda^4}} \right) \\ \tilde{V}_\lambda^{inst} &= 4e^2 \, \delta_{s_1 \bar{s}_3} \, \delta_{s_2 \bar{s}_4} \\ \tilde{V}_\lambda^{PT} &= e^2 N_{2,\lambda} \frac{1}{M_N^2} \mathrm{e}^{-\frac{M_0^4 + M_0'^4}{\lambda^4}} \ , \end{aligned} \tag{164}$$

where in the light-front frame, see eq. (261) and following in Appendix B

$$\begin{aligned} N_{1,\lambda} &= \delta_{s_1 s_3} \delta_{s_2 s_4} T_1^\perp \cdot T_2^\perp - \delta_{s_1 \bar{s}_2} \delta_{s_1 \bar{s}_3} \delta_{s_2 \bar{s}_4} 2m^2 \frac{(x-x')^2}{x x' (1-x)(1-x')} \\ &\quad + im\sqrt{2}(x'-x) \left[\delta_{s_1 \bar{s}_3} \delta_{s_2 s_4} \frac{s_1}{x x'} T_1^\perp \cdot \varepsilon_{s_1}^\perp + \delta_{s_1 s_3} \delta_{s_2 \bar{s}_4} \frac{s_2}{(1-x)(1-x')} T_2^\perp \cdot \varepsilon_{s_2}^\perp \right] \\ N_{2,\lambda} &= \delta_{s_1 \bar{s}_2} \delta_{s_3 \bar{s}_4} T_3^\perp \cdot T_4^\perp + \delta_{s_1 s_2} \delta_{s_3 s_4} \delta_{s_1 s_3} 2m^2 \frac{1}{x x' (1-x)(1-x')} \\ &\quad + im\sqrt{2} \left[\delta_{s_3 \bar{s}_4} \delta_{s_1 s_2} \frac{s_1}{x(1-x)} T_3^\perp \cdot \varepsilon_{s_1}^\perp - \delta_{s_3 s_4} \delta_{s_1 \bar{s}_2} \frac{s_3}{x'(1-x')} T_4^\perp \cdot \varepsilon_{s_4}^{\perp *} \right] \\ &\quad \varepsilon_s^i = -\frac{1}{\sqrt{2}} (s, i) \end{aligned} \tag{165}$$

and

$$\begin{aligned} T_1^i &= -\left[2\frac{(k_\perp - k'_\perp)^i}{(x-x')} + \frac{k_\perp^i(s_2)}{(1-x)} + \frac{k_\perp'^i(\bar{s}_2)}{(1-x')} \right] ; \qquad T_2^i = 2\frac{(k_\perp - k'_\perp)^i}{(x-x')} - \frac{k_\perp^i(s_1)}{x} - \frac{k_\perp'^i(\bar{s}_1)}{x'} \\ T_3^i &= -\frac{k_\perp'^i(\bar{s}_3)}{x'} + \frac{k_\perp'^i(s_3)}{(1-x')} ; \qquad T_4^i = \frac{k_\perp^i(\bar{s}_1)}{(1-x)} - \frac{k_\perp^i(s_1)}{x} \\ & k_\perp^i(s) = k_\perp^i + is\varepsilon_{ij} k_\perp^j ; \qquad \varepsilon_{ij} = \varepsilon_{ij3} ; \qquad \bar{s} = -s \end{aligned}$$

with the definitions

$$\begin{aligned}
\tilde{\Delta}_1 &= \frac{(xk'_\perp - x'k_\perp)^2 + m^2(x-x')^2}{xx'}\ ; \qquad \tilde{\Delta}_2 = \Delta_1|_{x\to(1-x),x'\to(1-x')} \\
&\Delta_1 = \frac{\tilde{\Delta}_1}{x'-x}\ ; \qquad \Delta_2 = \frac{\tilde{\Delta}_2}{x'-x} \\
\tilde{\Delta}_3 &= (k_\perp - k'_\perp)^2 + \frac{1}{2}(x-x')A + |x-x'|\left(\frac{1}{2}(M_0^2 + M_0'^2) - M_N^2\right) \\
&M_0^2 = \frac{k_\perp^2 + m^2}{x(1-x)}\ ; \qquad M_0'^2 = \frac{k_\perp'^2 + m^2}{x'(1-x')} \\
&A = (k_\perp^2 + m^2)\left(\frac{1}{1-x} - \frac{1}{x}\right) + (k_\perp'^2 + m^2)\left(\frac{1}{x'} - \frac{1}{1-x'}\right) \\
&P^- = \frac{(P^\perp)^2 + M_N^2}{P^+}\ ; \qquad P = (P^+, P^\perp)\ ; \qquad M_N = 2m + B_N\ . \qquad (166)
\end{aligned}$$

Note, that the rescaled potential, eq. (162), does not depend on the total momentum P^+, i.e. is invariant under light-front boosts.

Second, we use the **instant frame**. We rewrite the electron-positron interaction in both exchange and annihilation channels as follows

$$\begin{aligned}
\tilde{V} &= \tilde{V}_{exch} + \tilde{V}_{ann} \\
&= -e^2 N_{1,\lambda}\left[\left(\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\tilde{\Delta}_1^2 + \tilde{\Delta}_2^2}\right)(1 - \mathrm{e}^{-\frac{\Delta_1^2+\Delta_2^2}{\lambda^4}})c_{ex}^{gen} + \frac{1}{\tilde{\Delta}_3}\mathrm{e}^{-\frac{\Delta_1^2+\Delta_2^2}{\lambda^4}}c_{ex}^{PT}\right] \\
&+ \left(-\frac{4e^2}{(x-x')^2}\,\delta_{s_1 s_3}\,\delta_{s_2 s_4}\right)c_{ex}^{inst} \\
&+ e^2 N_{2,\lambda}\left[\left(\frac{M_0^2 + M_0'^2}{M_0^4 + M_0'^4}\right)(1 - \mathrm{e}^{-\frac{M_0^4+M_0'^4}{\lambda^4}})c_{an}^{gen} + \frac{1}{M_N^2}\mathrm{e}^{-\frac{M_0^4+M_0'^4}{\lambda^4}}c_{an}^{PT}\right] \\
&+ \left(4e^2\,\delta_{s_1\bar{s}_3}\,\delta_{s_2\bar{s}_4}\right)c_{an}^{inst}\ , \qquad (167)
\end{aligned}$$

where one has in the instant frame

$$\begin{aligned}
&x = \frac{1}{2}\left(1 + \frac{k_z}{\sqrt{\vec{p}^2 + m^2}}\right)\ ; \qquad x' = \frac{1}{2}\left(1 + \frac{k'_z}{\sqrt{\vec{p}^2 + m^2}}\right) \\
&M_0^2 = 4(\vec{p}^2 + m^2)\ ; \qquad M_0'^2 = 4(\vec{p}'^2 + m^2)\ ; \qquad (168)
\end{aligned}$$

and for all the quantities, defined in eqs. (165) and (166), the substitution $x(k_z), x'(k'_z)$ is to be done. The symbols c_{ex}^{gen} and others are introduced to indicate the origin of the different terms (here generated interaction coming from the exchange channel); all $c = 1$.

The expression eq. (167) has the physical interpretation. First notice, that Δ_1 and Δ_2 descibe the energy differences (energy transfer) in two corresponding $ee\gamma$-vertices appearing in the electron-positron interaction. Then the generated interaction (c_{ex}^{gen}) contributes mainly hard photon exchanges $\frac{\Delta_1}{\lambda^2} \sim \frac{\Delta_2}{\lambda^2} >> 1$, while the term arising from perturbative theory c_{ex}^{PT} gives rise to soft photons. Though the effective electron-positron interaction generally describes low energy physics, namely the renormalized effective Hamiltonian is constructed to eliminate perturbative (relativistic) photon exchanges, the information on the high energy sector is accumulated in the generated interaction. This makes possible to interpolate between two sectors, i.e. the sum of both terms in eq. (167) recovers the whole range of photon energies.

In the next section we analyse the general properties of the effective electron-positron interaction, generated by the unitary transformation in MSR scheme.

4.3.2 Electron-positron interaction in MSR

We give here the general analyses of the electron-positron interaction, obtained in MSR scheme. Basing on this analyses, we calculate in the next section positronium fine structure.

We summarize the interactions in $|e\bar{e}>$ sector, generated by the unitary transformation of MSR (section 3.3.2), in exchange channel fig. (8)

$$\begin{aligned}
\tilde{V}_\lambda^{gen} &= -e^2 N_{1,\lambda}\left(\frac{\int_\lambda^\infty \frac{df_{1\lambda'}}{d\lambda'} f_{2\lambda'} d\lambda'}{\tilde{\Delta}_1} + \frac{\int_\lambda^\infty \frac{df_{2\lambda'}}{d\lambda'} f_{1\lambda'} d\lambda'}{\tilde{\Delta}_2}\right) \\
\tilde{V}_\lambda^{PT} &= -e^2 N_{1,\lambda}\frac{1}{\tilde{\Delta}_3} f_{1\lambda} f_{2\lambda} \\
\tilde{V}_\lambda^{inst} &= -\frac{4e^2}{(x-x')^2}\delta_{s_1 s_3}\delta_{s_2 s_4}\ ,
\end{aligned} \tag{169}$$

where the similarity functions

$$f_{1\lambda} = f_\lambda(\Delta_1), f_{2\lambda} = f_\lambda(\Delta_2)\ , \tag{170}$$

are specified in eqs. (124) and (125) section 3.3.2; the energy denominators $\tilde{\Delta}_i, i = 1,2,3$ together with $N_{1,\lambda}$ are given in eqs. (165) and (166). Introduce the following notations

$$\begin{aligned}
\theta_1 &= \int_\lambda^\infty \frac{df_{1\lambda'}}{d\lambda'} f_{2\lambda'} d\lambda' \\
\theta_2 &= \int_\lambda^\infty \frac{df_{2\lambda'}}{d\lambda'} f_{1\lambda'} d\lambda'\ ,
\end{aligned} \tag{171}$$

Then the generated interaction is written

$$\tilde{V}_\lambda^{gen} = -e^2 N_{1,\lambda}\left(\frac{1}{2}(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2})(1-f_{1\lambda}f_{2\lambda}) + \frac{1}{2}(\frac{1}{\tilde{\Delta}_1}-\frac{1}{\tilde{\Delta}_2})(\theta_1-\theta_2)\right)\ , \tag{172}$$

where we have used the identities

$$\begin{aligned}
\frac{\theta_1}{\tilde{\Delta}_1}+\frac{\theta_2}{\tilde{\Delta}_2} &= \frac{1}{2}(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2})(\theta_1+\theta_2) + \frac{1}{2}(\frac{1}{\tilde{\Delta}_1}-\frac{1}{\tilde{\Delta}_2})(\theta_1-\theta_2) \\
\theta_1+\theta_2 &= 1-f_{1\lambda}f_{2\lambda}\ ,
\end{aligned} \tag{173}$$

The explicit form of similarity functions together with θ_i factors is

for Wegners flow equations

$$\begin{aligned}
f_\lambda(\Delta_1) &= \mathrm{e}^{-\frac{\Delta_1^2}{\lambda^4}} \\
\theta_1 &= \frac{\tilde{\Delta}_1^2}{\tilde{\Delta}_1^2+\tilde{\Delta}_2^2}(1-f_\lambda(\Delta_1)f_\lambda(\Delta_2))\ ,
\end{aligned} \tag{174}$$

for the similarity transformation of Glazek,Wilson

$$\begin{aligned}
f_\lambda(\Delta_1) &= \theta(\lambda^2-|\Delta_1|) \\
\theta_1 &= \theta(|\Delta_1|-|\Delta_2|)\theta(|\Delta_1|-\lambda^2) = \theta(\tilde{\Delta}_1-\tilde{\Delta}_2)\theta(|\Delta_1|-\lambda^2)\ ,
\end{aligned} \tag{175}$$

where

$$\Delta_1 = \frac{\tilde{\Delta}_1}{x'-x}, \Delta_2 = \frac{\tilde{\Delta}_2}{x'-x}\ , \tag{176}$$

We stress once more, that the unitary transformation of MSR is constructed in a way to avoid small energy denominators. Namely, the situation $\Delta_1 = \Delta_2 = 0$, corresponding to the process with no momentum transfer ($\tilde{\Delta}_1 = \tilde{\Delta}_2 = 0$ means in light front coordinates $x = x', k_\perp = k'_\perp$, i.e. $q = p_1 - p_3 = 0$), is excluded from the generated interaction for any finite $\lambda > 0$. Further we assume for all positive λ

$$\tilde{V}^{gen}_\lambda(q = 0) = 0, \forall \lambda \geq 0 \, , \tag{177}$$

This means also, that the diagonal matrix elements, left after the limit $\lambda \to 0$ is performed in the 'rest' sector to the first order in coupling ($ee\gamma$ vertex when $\tilde{\Delta}_1 = 0, \tilde{\Delta}_2 = 0$), do not contribute to the second order generated interaction in 'diagonal' sector. Therefore one can assume, that the 'rest' sector is completly eliminated by the unitary transformation of MSR.

In this limit ($\lambda \to 0$) the perturbative part of electron-positron interaction is zero

$$\tilde{V}^{PT} = 0 \, , \tag{178}$$

and electron-positron interaction is defined by the instantaneous interaction eq. (169) and the generated term at $\lambda \to 0$

$$\begin{aligned} \tilde{V}^{gen}_{\lambda\to 0} &= -e^2 N_{1,\lambda} \left(\frac{\int_0^\infty \frac{df_{1\lambda'}}{d\lambda'} f_{2\lambda'} d\lambda'}{\tilde{\Delta}_1} + \frac{\int_0^\infty \frac{df_{2\lambda'}}{d\lambda'} f_{1\lambda'} d\lambda'}{\tilde{\Delta}_2} \right) \\ \tilde{V}^{inst}_{\lambda\to 0} &= -\frac{4e^2}{(x-x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \, , \end{aligned} \tag{179}$$

The expression

$$\tilde{V}^{|e\bar{e}>} = \tilde{V}^{gen}_{\lambda\to 0} + \tilde{V}^{inst}_{\lambda\to 0} \, , \tag{180}$$

gives the effective electron-positron interaction obtained in MSR. Note, that when $\lambda \to 0$ no difficulties (compared with SR) on the convergency of unitary transformation (MSR scheme) and (or) convergency of perturbative theory occure. This is closely related with the maintenance of the rotational invariance and gauge invariance of perturbative calculations.

The explicit form of generated interaction when $\lambda \to 0$ is given in the case of

flow equations

$$\tilde{V}^{gen}_{\lambda\to 0} = -e^2 N_1 \left(\frac{\tilde{\Delta}_1 + \tilde{\Delta}_2}{\tilde{\Delta}_1^2 + \tilde{\Delta}_2^2} \right) \, , \tag{181}$$

similarity transformation

$$\tilde{V}^{gen}_{\lambda\to 0} = -e^2 N_1 \left(\frac{\theta(\tilde{\Delta}_1 - \tilde{\Delta}_2)}{\tilde{\Delta}_1} + \frac{\theta(\tilde{\Delta}_2 - \tilde{\Delta}_1)}{\tilde{\Delta}_2} \right) \, , \tag{182}$$

We consider now several special cases to analyse the general expression eq. (169).

I. Consider the **energy conserving process** (EC), i.e. when the following condition is valid

$$p_1^- + p_2^- = p_3^- + p_4^- \, , \tag{183}$$

for the light front coordinates of the process fig. (8). This gives rise to

$$\begin{aligned} \Delta_1 &= \Delta_2 \\ f_\lambda(\Delta_1) &= f_\lambda(\Delta_2) \, , \end{aligned} \tag{184}$$

Making use of the formula eq. (172), one has for the generated interaction

$$\tilde{V}_\lambda^{gen}|_{EC} = -e^2 N_{1,\lambda}\frac{1}{2}(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2})(1-f_{1\lambda}f_{2\lambda}) = -e^2 N_{1,\lambda}\frac{1}{\tilde{\Delta}_1}(1-f_{1\lambda}^2)\ , \tag{185}$$

and for the perturbative part

$$\tilde{V}_\lambda^{PT}|_{EC} = -e^2 N_{1,\lambda}\frac{1}{\tilde{\Delta}_3}f_{1\lambda}^2\ , \tag{186}$$

where

$$f_{1\lambda} = f_\lambda(\Delta_1) \tag{187}$$

When the following condition holds

$$M_N^2 = \frac{1}{2}(M_0^2 + M_0'^2)\ , \tag{188}$$

one has for the energy conserving process ($\tilde{\Delta}_1 = \tilde{\Delta}_2$)

$$\frac{\tilde{\Delta}_1+\tilde{\Delta}_2}{2} = \tilde{\Delta}_3\ , \tag{189}$$

This gives rise to

$$\begin{aligned}\tilde{V}_\lambda^{gen}|_{EC} &= -e^2 N_{1,\lambda}\frac{1}{\tilde{\Delta}_3}(1-f_\lambda^2(\Delta_3))\\ \tilde{V}_\lambda^{PT}|_{EC} &= -e^2 N_{1,\lambda}\frac{1}{\tilde{\Delta}_3}f_\lambda^2(\Delta_3)\ ,\end{aligned} \tag{190}$$

where we have introduced

$$\Delta_3 = \frac{\tilde{\Delta}_3}{x'-x}\ , \tag{191}$$

Expression eq. (190) shows clearly the sense of the unitary transformation in MSR scheme: how the elimination of $ee\gamma$ vertex in the 'rest' sector with the flow parameter l (or cutoff λ) generates the new interaction for the $|e\bar{e}>$ Fock component in 'diagonal' sector. For the energy conserving process together with the condition eq. (188) the resulting generated interaction (at $\lambda \to 0$) is given by the pure one-photon exchange diagram (i.e. by the initial perturbative interaction at $\lambda = \Lambda \to \infty$)

$$\tilde{V}_{\lambda\to 0}^{gen}|_{EC} = \tilde{V}_{\lambda=\Lambda\to\infty}^{PT}|_{EC}\ . \tag{192}$$

II.We consider the **collinear limit**, i.e. when $|x-x'| \to 0$, and show, that though the instantaneous term is singular in this limit the whole electron-positron interaction is finite. Namely the generated to the second order interaction insures the absence of collinear divergencies.

For this purpose we rewrite the electron-positron interaction eq. (169) as follows

$$\begin{aligned}&\tilde{V}_\lambda^{|e\bar{e}>} = \tilde{V}_\lambda^{gen} + \tilde{V}_\lambda^{PT} + \tilde{V}_\lambda^{inst}\\ &= -e^2 N_{1,\lambda}\frac{1}{\tilde{\Delta}_3} - \frac{4e^2}{(x-x')^2}\delta_{s_1s_3}\delta_{s_2s_4}\\ &-e^2 N_{1,\lambda}\left(\frac{1}{2}(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2}) - \frac{1}{\tilde{\Delta}_3}\right)(1-f_{1\lambda}f_{2\lambda})\\ &-e^2 N_{1,\lambda}\frac{1}{2}(\frac{1}{\tilde{\Delta}_1}-\frac{1}{\tilde{\Delta}_2})(\theta_1-\theta_2)\ ,\end{aligned} \tag{193}$$

The first two terms (perturbative photon exchange and instantaneous interaction) define the initial value of electron-positron interaction ($\lambda = \Lambda \to \infty$), the next terms are generated by the unitary transformation and describe the change of electron-positron interaction with λ.

In the collinear limit $|x - x'| \to 0$ the λ-dependent terms (namely similarity functions $f_{i\lambda} = f_\lambda(\frac{\Delta_i}{\lambda})$, $\Delta_i = \frac{\tilde{\Delta}_i}{x'-x}$ and θ_i factors, $i = 1, 2$) are given at the value $\lambda = 0$.

Since the singular contribution is carried by the instantaneous interaction we consider the whole interaction only in diagonal spin sector, i.e. $\delta_{s_1 s_3}\delta_{s_2 s_4}$.

The first two terms can be written

$$-e^2 N_{1,\lambda}(s_1 s_2 \to s_3 s_4)\frac{1}{\tilde{\Delta}_3} - \frac{4e^2}{(x-x')^2}\delta_{s_1 s_3}\delta_{s_2 s_4}$$
$$= -\frac{2e^2}{\tilde{\Delta}_3}(\frac{1}{2}N_{1,\lambda}(s_1 s_2 \to s_3 s_4) + \tilde{\Delta}_3\frac{2}{(x-x')^2}\delta_{s_1 s_3}\delta_{s_2 s_4}) \ , \qquad (194)$$

where $N_{1,\lambda}$, defined in eq. (165), is given

$$\frac{1}{2}N_{1,\lambda}(s_1 s_2 \to s_3 s_4) = -2\frac{(\vec{k}_\perp - \vec{k}'_\perp)^2}{(x-x')^2} - \frac{1}{(x-x')}\left(k_\perp^2(\frac{1}{1-x} - \frac{1}{x}) - k_\perp'^2(\frac{1}{1-x'} - \frac{1}{x'})\right)$$
$$+ f(s_1 s_2 \to s_3 s_4)$$
$$f(++ \to ++) = \frac{k_\perp k'_\perp}{xx'(1-x)(1-x')}\mathrm{e}^{-i(\phi-\phi')}$$
$$f(+- \to +-) = k_\perp^2\frac{1}{x(1-x)} + k_\perp'^2\frac{1}{x'(1-x')}$$
$$+ k_\perp k'_\perp\left(\frac{1}{xx'}\mathrm{e}^{i(\phi-\phi')} + \frac{1}{(1-x)(1-x')}\mathrm{e}^{-i(\phi-\phi')}\right)$$
$$\vec{k}_\perp = k_\perp(\cos\phi, \sin\phi) \ , \qquad (195)$$

We have used the notations '+'-spin up, '−'-spin down.

Assuming the condition

$$M_N^2 = \frac{1}{2}(M_0^2 + M_0'^2) \ , \qquad (196)$$

one has

$$\tilde{\Delta}_3\frac{2}{(x-x')^2} = 2\frac{(\vec{k}_\perp - \vec{k}'_\perp)^2}{(x-x')^2} + \frac{1}{(x-x')}(k_\perp^2(\frac{1}{1-x} - \frac{1}{x}) - k_\perp'^2(\frac{1}{1-x'} - \frac{1}{x'}))$$
$$+ m^2(\frac{1}{xx'} + \frac{1}{(1-x)(1-x')}) \ , \qquad (197)$$

Combining all terms together, we get

$$-e^2 N_{1,\lambda}(s_1 s_2 \to s_3 s_4)\frac{1}{\tilde{\Delta}_3} - \frac{4e^2}{(x-x')^2}\delta_{s_1 s_3}\delta_{s_2 s_4} =$$
$$-\frac{2e^2}{\tilde{\Delta}_3}\left(m^2(\frac{1}{xx'} + \frac{1}{(1-x)(1-x')}) + f(s_1 s_2 \to s_3 s_4)\right) \ , \qquad (198)$$

This part is finite, i.e. collinear singularity in the instantaneous interaction is cancelled exactly by the perturbative part. To the leading order in $\delta x = x - x'$ one has

$$-e^2 N_{1,\lambda}(s_1 s_2 \to s_3 s_4)\frac{1}{\tilde{\Delta}_3} - \frac{4e^2}{(x-x')^2}\delta_{s_1 s_3}\delta_{s_2 s_4} =$$
$$-\frac{2e^2}{(\vec{k}_\perp - \vec{k}'_\perp)^2}\left(m^2(\frac{1}{x^2} + \frac{1}{(1-x)^2}) + f|_{x=x'}(s_1 s_2 \to s_3 s_4)\right) + O(\delta x) \ , \qquad (199)$$

For the next term we perform the expansion with respect to δx

$$\frac{1}{2}(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2})-\frac{1}{\tilde{\Delta}_3}=\frac{(\vec{k}_\perp+\vec{k}'_\perp)^2}{4(\vec{k}_\perp-\vec{k}'_\perp)^4}\frac{\delta x^2}{x^2(1-x)^2}+O(\delta x^3)\ , \tag{200}$$

then one has

$$-e^2N_{1,\lambda}(s_1s_2\to s_3s_4)\left(\frac{1}{2}(\frac{1}{\tilde{\Delta}_1}+\frac{1}{\tilde{\Delta}_2})-\frac{1}{\tilde{\Delta}_3}\right)(1-f_{1\lambda}f_{2\lambda})=$$
$$-\frac{2e^2}{(\vec{k}_\perp-\vec{k}'_\perp)^2}\left(-\frac{(\vec{k}_\perp+\vec{k}'_\perp)^2}{2}\frac{1}{x^2(1-x)^2}\right)+O(\delta x)\ , \tag{201}$$

Both expressions eq. (199),eq. (201) do not depend on the explicit form of unitary transformation performed. The last term does depend in collinear limit on the explicit form of similarity function.

In the collinear limit θ_i-factors eq. (174),eq. (175) are given in the case of flow equations

$$\theta_1|_{x=x'}=\frac{\tilde{\Delta}_1^2}{\tilde{\Delta}_1^2+\tilde{\Delta}_2^2}\ , \tag{202}$$

and similarity transformation

$$\theta_1|_{x=x'}=\theta(\tilde{\Delta}_1-\tilde{\Delta}_2)\ , \tag{203}$$

Performing the expansion in δx, one has in both cases corresponding

$$-e^2N_{1,\lambda}(s_1s_2\to s_3s_4)\frac{1}{2}\left(\frac{1}{\tilde{\Delta}_1}-\frac{1}{\tilde{\Delta}_2}\right)(\theta_1-\theta_2)=\begin{cases}-\frac{2e^2}{(\vec{k}_\perp-\vec{k}'_\perp)^2}\left((\vec{k}_\perp+\vec{k}'_\perp)^2\frac{1}{x^2(1-x)^2}\right), \\ flow\quad equations \\ \\ \sim 0, similarity\quad transformation\end{cases} \tag{204}$$

Combining all terms, one has the following electron-positron interaction in collinear limit (considered only the spin sector $\delta_{s_1s_3}\delta_{s_2s_4}$)

$$\tilde{V}_\lambda^{|e\bar{e}>}(x=x')=\begin{cases}-\frac{2e^2}{(\vec{k}_\perp-\vec{k}'_\perp)^2}\left(m^2(\frac{1}{x^2}+\frac{1}{(1-x)^2})+\frac{(\vec{k}_\perp+\vec{k}'_\perp)^2}{2}\frac{1}{x^2(1-x)^2}+f|_{x=x'}(s_1s_2\to s_3s_4)\right), \\ flow\quad equations \\ \\ -\frac{2e^2}{(\vec{k}_\perp-\vec{k}'_\perp)^2}\left(m^2(\frac{1}{x^2}+\frac{1}{(1-x)^2})-\frac{(\vec{k}_\perp+\vec{k}'_\perp)^2}{2}\frac{1}{x^2(1-x)^2}+f|_{x=x'}(s_1s_2\to s_3s_4)\right), \\ similarity\quad transformation\end{cases} \tag{205}$$

that is finite.

III.Let us analyse **what kind of interaction** (repulsive or attractive) arise from the effective Hamiltonian in the electron-positron sector eq. (169).

First note, that in the light-front frame the generated interaction has definite sign for any value of λ. Namely the term in bracket of generated interaction eq. (169) is always positive, since energy denominators $\tilde{\Delta}_1,\tilde{\Delta}_2$ are positive on the light front, and $\frac{df_{ij}}{d\lambda}\geq 0, f_{ij}\geq 0$ ($f_{ij}(\lambda)$ has the same behavior as $u_{ij}(\lambda)$, defined in Appendix A).

When $\lambda=0$ generated interaction together with instantaneous term give rise to the attractive electron-positron interaction in the whole parameter space.

To the leading order of nonrelativistic approximation

$$\frac{|\vec{p}|}{m} \ll 1 \ , \tag{206}$$

one has in the instant coordinate frame

$$\begin{aligned}
&\tilde{\Delta}_1 \sim \tilde{\Delta}_2 \sim \tilde{\Delta}_3 = \tilde{\Delta} = (\vec{p} - \vec{p'})^2 \\
&\tilde{V}_\lambda^{gen} \approx -e^2 \frac{N_1}{(\vec{p} - \vec{p'})^2} (1 - f_\lambda^2(\Delta)) \\
&\tilde{V}_\lambda^{PT} \approx -e^2 \frac{N_1}{(\vec{p} - \vec{p'})^2} f_\lambda^2(\Delta) \\
&\Delta = \frac{(\vec{p} - \vec{p'})^2}{x' - x} \ ,
\end{aligned} \tag{207}$$

This gives for the electron-positron interaction in the whole nonrelativistic range of λ

$$\begin{aligned}
&\lambda \ll m \\
&\tilde{V}^{|e\bar{e}>} \approx -e^2 \frac{N_1}{(\vec{p} - \vec{p'})^2} - \frac{4e^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \ ,
\end{aligned} \tag{208}$$

the λ-independent result. Making use of the following expressions

$$\begin{aligned}
&N_1^{diag} \approx -4 \frac{(\vec{k}_\perp - \vec{k}'_\perp)^2}{(x - x')^2} \delta_{s_1 s_3} \delta_{s_2 s_4} \\
&(\vec{p} - \vec{p'})^2 = (\vec{k}_\perp - \vec{k}'_\perp)^2 + (k_z - k'_z)^2 \approx (\vec{k}_\perp - \vec{k}'_\perp)^2 + 4m^2 (x - x')^2 \ ,
\end{aligned} \tag{209}$$

one get to the leading order of nonrelativistic approximation the 3-d Coulomb interaction

$$\tilde{V}^{|e\bar{e}>} \approx 16m^2 \left(-\frac{e^2}{(\vec{p} - \vec{p'})^2} \right) \delta_{s_1 s_3} \delta_{s_2 s_4} \ ; \tag{210}$$

hence the rotational invariance is restored to this order. This result eq. (210) is valid for any nonrelativistic value of λ, and for any unitary transformation in MSR (i.e. $\forall f_{ij}$) performed.

One can interprete the independence of the interaction eq. (208) on λ, when λ is in nonrelativistic domain $\lambda \ll m$. In this region all photons, that give rise to perturbative (relativistic) corrections, are almost eliminated. Therefore effectively only the lowest Fock state ($|e\bar{e}>$) contribute to the interaction and trancation of the whole Fock space to the lowest Fock component is valid in this case.

The explicit λ-dependence (through the similarity functions) in the interaction signals to the presence of the terms in effective Hamiltonian that mix different Fock component, i.e. particle number changing interactions. Then trancation is not possible.

One way out is to use different approximations, resulting in freezing effectively the contribution of particle number changing (Fock state changing) interactions (nonrelativistic approximation [6]; the definite choice of scale (size of the band in SR scheme) λ, depending on the system considered [11]).

In the case of MSR the effective Hamiltonian is defined in the limit $\lambda \to 0$, where all Fock states are completly decoupled and one is able to solve the bound state problem separately in each 'diagonal' sector.

In the next section, making use of the nonrelativistic approximation for the electron (positron) momentum

$$\frac{|\vec{p}|}{m} = O(\alpha) < 1 \ , \tag{211}$$

in the effective electron-positron interaction eq. (167), obtained by flow equations, we calculate the positronium splitting analytically.

4.4 Positronium's fine structure and rotational invariance

In the nonrelativistic approximation we obtain the following effective electron-positron interaction, eq. (167),

$$\begin{aligned}
V'_\lambda &= \frac{1}{2(2\pi)^3}\frac{1}{4m}\frac{1}{2m}\left(1-\frac{\vec{p}^2}{2m^2}\right)\tilde{V}_\lambda \\
\tilde{V}_\lambda &= \tilde{V}^{exch}_\lambda + \tilde{V}^{ann}_\lambda \\
&= -\frac{e^2 N_1}{(\vec{p}-\vec{p'})^2}\left(1-\mathrm{e}^{-2\left(\frac{\Delta}{\lambda^2}\right)^2}\right) c^{gen}_{ex} \; - \frac{e^2 N_1}{(\vec{p}-\vec{p'})^2+|x-x'|(M_0^2-M_N^2)}\,\mathrm{e}^{-2\left(\frac{\Delta}{\lambda^2}\right)^2} c^{PT}_{ex} \\
&\quad - \frac{4e^2}{(x-x')^2}\,\delta_{s_1 s_3}\,\delta_{s_2 s_4}\,c^{inst}_{ex} \\
&\quad + \frac{e^2 N_2}{4m^2}\left(1-\mathrm{e}^{-2\left(\frac{4m^2}{\lambda^2}\right)^2}\right) c^{gen}_{an} + \frac{e^2 N_2}{M_N^2}\,\mathrm{e}^{-2\left(\frac{4m^2}{\lambda^2}\right)^2}\, c^{PT}_{an} \\
&\quad + 4e^2\,\delta_{s_1\bar{s}_3}\,\delta_{s_2\bar{s}_4}\,c^{inst}_{an} \ ,
\end{aligned} \tag{212}$$

where the energy denominators and exponential factors were simplified using

$$\begin{aligned}
&x-x' = \frac{k_z-k'_z}{2m}\left[1+\frac{\vec{p}^2}{2m^2}\right] + O\left(m^2\left(\frac{p}{m}\right)^5\right) \\
&\tilde{\Delta}_1 = \tilde{\Delta}_2 = (\vec{p}-\vec{p'})^2 + O\left(m^2\left(\frac{p}{m}\right)^5\right) \\
&\tilde{\Delta}_3 = (\vec{p}-\vec{p'})^2 + |x-x'|(M_0^2-M_N^2) + O\left(m^2\left(\frac{p}{m}\right)^4\right) \\
&\Delta_1 = \Delta_2 = \frac{2m(\vec{p'}-\vec{p})^2}{(k'_z-k_z)}\left[1+O\left(\left(\frac{p}{m}\right)^2\right)\right] \ ; \qquad \Delta = \frac{2m(\vec{p'}-\vec{p})^2}{(k'_z-k_z)} \\
&M_0^2 = 4m^2 + O\left(m^2\left(\frac{p}{m}\right)^2\right) \\
&M_N^2 = 4m^2 + 4mB_N + O\left(m^2\left(\frac{B_N}{m}\right)^2\right) = 4m^2 + 4mB_N^{(0)} \ ,
\end{aligned} \tag{213}$$

and the explicit expression of Jacobian for the coordinate change is

$$\sqrt{J(p)J(p')} = \frac{1}{2m}\left[1-\frac{\vec{p}^2}{2m^2}+O\left(\frac{k_z^2}{m^2},\frac{k_z'^2}{m^2}\right)\right] \ , \tag{214}$$

having introduced the leading order binding energy $B_N^{(0)}$. Making use of the nonrelativistic approximation $B_N^{(0)}/m \ll 1$ we have for the interaction

$$V'_\lambda = \frac{1}{2(2\pi)^3}\frac{1}{4m}\frac{1}{2m}\left(1-\frac{\vec{p}^2}{2m^2}\right)$$

$$\times\Big[\Big(-\frac{e^2N_1}{(\vec{p}-\vec{p'})^2}\,(c_{ex}^{gen},c_{ex}^{PT})\;-\frac{16e^2m^2}{(k_z-k_z')^2}\left(1+\frac{\vec{p}^2}{m^2}\right)\,c_{ex}^{inst}\,\delta_{s_1s_3}\,\delta_{s_2s_4}$$
$$+\frac{e^2N_2}{4m^2}\,(c_{an}^{gen},c_{an}^{PT})\qquad\qquad+\;4e^2\,c_{an}^{inst}\,\delta_{s_1\bar{s}_2}\,\delta_{s_3\bar{s}_4}\Big)$$
$$+\Big(-\frac{e^2N_1}{(\vec{p}-\vec{p'})^2}\frac{4mB_N^{(0)}}{|\Delta|}\,c_{ex}^{PT}\,\mathrm{e}^{-(\frac{\Delta}{\lambda^2})^2}$$
$$-\frac{e^2N_2}{4m^2}\frac{B_N^{(0)}}{m}\,c_{an}^{PT}\,\mathrm{e}^{-\left(\frac{4m^2}{\lambda^2}\right)^2}\;\Big)\Big]\,, \tag{215}$$

where (c^{gen},c^{PT}) showes that both term, generated and perturbative interactions, contribute to corresponding term (we remember that all $c=1$).

The remarkable feature of the part of interaction standing in the first bracket is that it does not depend on the UV cutoff λ. The next term in the second bracket arises from the perturbative photon exchange and has the typical 'energy shell' structure for the relativistic effects, namely these terms are important when $\lambda >> m$. Further we calculate the ground state positronium mass and therefore restrict the cutoff to be in the nonrelativistic domain

$$\lambda \ll m\,, \tag{216}$$

where the second term in eq. (215) vanishes and we are left with the following form for the renormalized $e\bar{e}$ interaction in the nonrelativistic approximation:

$$V'(\vec{p},\vec{p'}) \;=\; \frac{1}{2(2\pi)^3}\frac{1}{4m}\frac{1}{2m}\left(1-\frac{\vec{p}^2}{2m^2}\right) \tag{217}$$
$$\times\Big[-\frac{e^2N_1}{(\vec{p}-\vec{p'})^2}\,(c_{ex}^{eff},c_{ex}^{PT})\;-\frac{16e^2m^2}{(k_z-k_z')^2}\left(1+\frac{\vec{p}^2}{m^2}\right)\,c_{ex}^{inst}\,\delta_{s_1s_3}\,\delta_{s_2s_4}$$
$$+\frac{e^2N_2}{4m^2}\,(c_{an}^{eff},c_{an}^{PT})\qquad\qquad+\;4e^2\,c_{an}^{inst}\,\delta_{s_1\bar{s}_2}\,\delta_{s_3\bar{s}_4}\Big]\,.$$

We perform the nonrelativistic expansion of the factors N_1 and N_2 appearing in the interaction. The term N_1 contributes in V' to the order
$\underline{O(1),O\left(\left(\frac{p}{m}\right)^2\right)}$:

$$-T_1^{\perp}T_2^{\perp} \;=\; 16m^2\frac{q_\perp^2}{q_z^2}\left(1+\frac{\vec{p}^2}{m^2}\right)+16\frac{q_\perp^i}{q_z}\left(k_\perp^i k_z+k_\perp'^i k_z'\right)$$
$$-16i(s_1+s_2)[k_\perp' k_\perp]-4(k_\perp+k_\perp')^2+4s_1s_2q_\perp^2\,,$$

$\underline{O\left(\frac{p}{m}\right),O\left(\left(\frac{p}{m}\right)^2\right)}$:

$$im\sqrt{2}(x'-x)\left(\frac{s_1}{xx'}\,\varepsilon_{s_1}^{\perp}\cdot T_1^{\perp}\,\delta_{\bar{s}_1s_3}\,\delta_{s_2s_4}\;+\frac{s_2}{(1-x)(1-x')}\,\varepsilon_{s_2}^{\perp}\cdot T_2^{\perp}\,\delta_{\bar{s}_4s_2}\,\delta_{s_1s_3}\right)$$
$$=8\,\delta_{\bar{s}_1s_3}\,\delta_{s_2s_4}\left[m\,(iq_\perp^x-s_1q_\perp^y)\left(1-\frac{k_z+k_z'}{m}\right)+q_z\,(i\tilde{p}_\perp^x-s_1\tilde{p}_\perp^y)+\frac{1}{2}s_2q_z(q_\perp^y-is_1q_\perp^x)\right]$$
$$-\,\delta_{\bar{s}_4s_2}\,\delta_{s_1s_3}\left[m\,(iq_\perp^x-s_2q_\perp^y)\left(1+\frac{k_z+k_z'}{m}\right)-q_z\,(i\tilde{p}_\perp^x-s_2\tilde{p}_\perp^y)-\frac{1}{2}s_1q_z(q_\perp^y-is_2q_\perp^x)\right]\,,$$

$\underline{O\left(\left(\frac{p}{m}\right)^2\right)}$:

$$2m^2\frac{(x-x')^2}{xx'(1-x)(1-x')} = 8q_z^2 \ .$$

While the term N_2 contributes to V' to the order
$\underline{O\left(\left(\frac{p}{m}\right)^2\right)}$:

$$2m^2\frac{1}{xx'(1-x)(1-x')} = 32m^2 \ . \tag{218}$$

In these formulas we have used $[k'_\perp, k_\perp] = \varepsilon_{ij}k'^i_\perp k^j_\perp$, $\varepsilon_{ij} = \varepsilon_{ij3}$ and $\varepsilon^i_s = -\frac{1}{\sqrt{2}}(s,i)$; also the following variables have been introduced

$$\begin{aligned} &q_\perp = k'_\perp - k_\perp \ , \qquad (\perp = x,y) \ , \ q_z = k'_z - k_z \\ &\tilde{p}_\perp = \frac{k_\perp + k'_\perp}{2} \ . \end{aligned} \tag{219}$$

We leave aside for the future work the analysis of the expressions for N_1 and N_2, where also in this form some terms can be identified as spin-orbit and spin-spin interactions in the transverse plane and in longitudinal (z) direction.

Instead we follow [6], where an analogous calculation of singlet-triplet ground state mass splitting of positronium was performed in the similarity scheme. This means, that we can, except for the leading order term $O(1)$, drop in N_1 the part diagonal in spin space. Also the terms of the type $f = k^{x,y}_\perp k_z$, $k^{x,y}_\perp k'_z$, $k^x_\perp k^y_\perp$ do not contribute to the ground state mass splitting, since

$$\int d^3p d^3p' \, \Phi^*_{100}(\vec{p}) \, \frac{f}{\vec{q}^2} \, \Phi_{100}(\vec{p'}) \ , \tag{220}$$

averaging over directions, gives zero.

We obtain for the $e\bar{e}$-potential to the **leading order** $O(1)$ of nonrelativistic expansion

$$\begin{aligned} V'^{(0)}(\vec{p'},\vec{p}) &= \frac{1}{2(2\pi)^3}\frac{1}{4m}\frac{1}{2m}\left(1-\frac{\vec{p}^2}{2m^2}\right) \\ &\quad \times\left[\frac{16e^2m^2}{\vec{q}^2}\frac{q^2_\perp}{q^2_z}\left(1+\frac{\vec{p}^2}{m^2}\right)\ (c^{gen}_{ex}, c^{PT}_{ex}) \ - \frac{16e^2m^2}{q^2_z}\ \left(1+\frac{\vec{p}^2}{m^2}\right)\ c^{inst}_{ex}\right]\delta_{s_1s_3}\,\delta_{s_2s_4} \\ &= -\frac{\alpha}{2\pi^2}\frac{1}{\vec{q}^2}\left(1+\frac{\vec{p}^2}{2m^2}\right)\delta_{s_1s_3}\,\delta_{s_2s_4} \\ &\longrightarrow \quad V(r)\left(1+\frac{\vec{p}^2}{2m^2}\right) \ . \end{aligned} \tag{221}$$

Remembering $\vec{q} = \vec{p'} - \vec{p}$, Fourier transformation to the coordinate space with respect to $\vec{q}$ has been performed in the last expression. To the leading order of NR expansion we have reproduced the Coulomb potential, defined before as the leading order of BSPT. Note, this is true for any UV cutoff within the nonrelativistic range $\lambda \ll m$.

We combine this expression with the kinetic term from the Schrödinger equation, eq. (155), and write it in the form

$$\frac{1}{m}\left(1+\frac{V(r)}{2m}\right)\vec{p}^2 + V(r) \ . \tag{222}$$

Here the potential $V(r)$ plays a different role in the two terms. In the first term, corresponding to kinetic energy, it generates an effective mass of the electron, which depends on the relative position and manifests the non-locality of the interaction. The second term is the usual potential energy, in our case, the Coulomb interaction.

The energy of the Coulomb level with quantum numbers (nlm) is standard

$$M_0^2 =< \Phi_{nlm}|V^{'(0)}|\Phi_{nlm} >= \int d^3p d^3p' \Phi^*_{nlm}(\vec{p})\, V^{'(0)}\, \Phi_{nlm}(\vec{p'}) = -\frac{m\alpha^2}{2n^2}\ , \tag{223}$$

where the Coulomb wave functions Φ_{nlm} were defined in eq. (157). We have used in eq. (223) the following representation

$$\begin{aligned}
&(\vec{p}-\vec{p'})^2 = \frac{(e_n^2+\vec{p}^2)\,(e_n^2+\vec{p'}^2)}{4e_n^2}\,(u-u')^2 \\
&\frac{1}{(u-u')^2} = \sum_\mu \frac{2\pi^2}{n}\, Y_\mu(\Omega_p)\, Y^*_\mu(\Omega_{p'}) \\
&d^3p = d\Omega_p \left(\frac{e_n^2+\vec{p}^2}{2e_n}\right)^3
\end{aligned} \tag{224}$$

and also orthogonality of the hyperspherical harmonics

$$\int d\Omega\, Y^*_\mu\, Y_{\mu'} = \delta_{\mu\mu'}\ . \tag{225}$$

More details can be found in [6].

The next to leading order $O\left(\frac{p}{m}\right)$

$$\begin{aligned}
\delta V^{(1)} \;=\; & \frac{1}{2(2\pi)^3}\frac{1}{4m}\frac{1}{2m}\left(-\frac{e^2}{\vec{q}^2}\right) \\
& \times (8m(iq^x_\perp - s_1 q^y_\perp)\delta_{s_1\bar{s}_3}\delta_{s_2 s_4} - 8m(iq^x_\perp - s_2 q^y_\perp)\delta_{s_1 s_3}\delta_{s_2\bar{s}_4})
\end{aligned} \tag{226}$$

contributes (because of the spin structure) to the second order of BSPT:

$$\delta M_2^2 = \sum_{\mu,s_i} \frac{< \Phi_{100}|\delta V^{(1)}|\Phi_{\mu,s_i} >< \Phi_{\mu,s_i}|\delta V^{(1)}|\Phi_{100} >}{M_1^2 - M_n^2}\ . \tag{227}$$

The order $O\left(\left(\frac{p}{m}\right)^2\right)$ (cf. remark after eq. (219)) is

$$\delta V^{(2)} = \frac{1}{2(2\pi)^3}\frac{1}{4m}\frac{1}{2m}\left(8e^2\frac{q_z^2}{\vec{q}^2}\delta_{s_1\bar{s}_2}\delta_{s_1\bar{s}_3}\delta_{s_2\bar{s}_4} + 8e^2\delta_{s_1 s_2}\delta_{s_3 s_4}\delta_{s_1 s_3} + 4e^2\delta_{s_1\bar{s}_3}\delta_{s_2\bar{s}_4}\right) \tag{228}$$

and contributes to the first order of BSPT:

$$\delta M_1^2 =< \Phi_{100}|\delta V^{(2)}|\Phi_{100} >\ . \tag{229}$$

Both contributions were calculated in [6] with the result

$$\begin{aligned}
\delta M^2 \;&=\; \delta M_1^2 + \delta M_2^2 \\
< 1|\delta M^2|1 > \;&=\; -\frac{5}{12}m\alpha^4 \\
< 2|\delta M^2|2 > \;&=\; < 3|\delta M^2|3 >=< 4|\delta M^2|4 >= \frac{1}{6}m\alpha^4\ ,
\end{aligned} \tag{230}$$

where the eigenvectors in spin space are defined as follows:

$$|1>=\frac{1}{\sqrt{2}}\left(|+->-|-+>\right),$$
$$|2>=\frac{1}{\sqrt{2}}\left(|+->+|-+>\right), \qquad |3>=|-->, \qquad |4>=|++> \ . \tag{231}$$

Making use of the relation between Coulomb energy units and Ryd $=\frac{1}{2}m\alpha^2$ we have the standard result for the singlet-triplet mass splitting for positronium, $\frac{7}{6}\alpha^2$Ryd. The degeneracy of the triplet ground state $n=1$ reflects the rotational invariance, manifest in the system in nonrelativistic approximation.

5 Conclusions and outlook

We proposed in this work the new Hamiltonian approach, the modified similarity renormalization (MSR) of Hamiltonians. By means of flow equations, acting in the energy space, the unitary transformation is performed, which aims to bring the field theoretical Hamiltonian to the block-diagonal form in Fock space.

The renormalized effective Hamiltonian, constructed in such a way, solves two problems. First, all 'diagonal' Fock sectors, i.e. the sectors belonging to Fock state conserving, are completly decoupled in the effective Hamiltonian. Second, elimination with the flow equations of 'rest' sectors, i.e. Fock state changing, perturbative order by order in coupling constant, enables to find to these orders all counterterms, corresponding to canonical operators (relevant and marginal) of the initial theory and to new operators, generated by flow equations. The 'new' counterterms are combinations of the 'canonical' one or can carry also new types of divergencies. This question we stayed aside in this work.

Summarizing, one is able to answer two questions: the physics of what Fock state and of what energy scale is descibed by the renormalized effective Hamiltonian.

Namely, one can pick out any block from 'diagonal' sector of effective Hamiltonian, acting on the definite Fock state, and solve separately for this block (Fock state) the corresponding physical problem. In particular, it is possible to trancate the block-diagonal effective Hamiltonian to the lowest Fock state (here discussed $|e\bar{e}>$), plug the corresponding interaction (from '$e\bar{e}$' block) into the Schrödinger equation, put it on the computer and solve the bound state problem (for positronium bound state) numericaly.

The problem that arises by numerical diagonalization of the lowest Fock sector matrix in the energy space (or by numerical solution of integral bound state equation, see main text) is the dependence of physical mass spectrum on the size of the matrix (or on UV cutoff imposed on the energies in the integral). This dependence is to be absorbed by including the corresponding counterterms, calculated analyticaly on the previous step. MSR scheme insures, that counterterms to add do not depend on the state (i.e. are the same for the lowest and next-to-lowest exited states) and on the Fock sector (i.e. trancation to the $|e\bar{e}>$ state do not prevent counterterms corresponding to the diagrams with more particles in intermediate state) [10]. This is due to the new interactions and corresponding counterterms, generated by the flow equations in MSR scheme.

In the work we have outlined the strategy to build the renormalized effective Hamiltonian, that can be fulfilled by means of flow equations in a systematic way (to the end $\lambda \to 0$) without knowing apriore any properties of the system considered further (i.e. no approximations are needed to make in the procedure). This is the main advantage of MSR scheme.

We remind, that through MSR scheme one simultaneously renormalizes in the energy space the initial field theoretical Hamitonian and constructs the effective Hamitonian, for which the Fock space trancation is valid. But rather the analytical solution of flow equations, as of any other iterrative method, demands to apply perturbative expansion in coupling constant. This means, that applicability of perturbative theory is closely related to the possibility of working in a trancated Fock space. Within the proposed approach one is able to improve sytematically this approximation.

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A Similarity renormalization

In this appendix we skip the idea of similarity renormalization (SR) proposed by Glazek and Wilson [2], the renormalization scheme of Hamiltonians by means of unitary transformation, and consider two alternative methods: the flow equations (FE) of Wegner [1] and similarity unitary transformation (SUT) [2], that are used in SR scheme and give rise to the renormalized Hamiltonian.

The similarity transformation aims to bring the field theoretical Hamiltonian to the most diagonal form, namely only the matrix elements between the free states with

$$|\Delta_{ij}| < \lambda \tag{232}$$

where $\Delta_{ij} = E_i - E_j$, are present in the renormalized Hamiltonian. For this purposes the continuous unitary transformation is performed to preserve unchanged the spectrum (eigenvalues) of the initial bare cutoff Hamiltonian. The demand of diagonal structure does not define completly the generator of the transformation. This freedom is used to eliminate small energy denominators in the final renormalized Hamiltonian. This results in a system of two self-consistent non-linear differential equations for the Hamiltonian $H(l)$ and the generator of the transformation $\eta(l)$. The dependence on the continuous flow parameter l in the flow equations by Wegner is replaced by the cutoff dependence λ in the similarity unitary transformation, with the connection

$$l = 1/\lambda^2 \tag{233}$$

in the renormalized Hamiltonians.

The difference of the two methods (FE) and (SUT) consists of the residual freedom in the choice of the direction of the infinitesimal rotation, actually defining how fast the non-diagonal matrix elements vanish.

We summarize the equations for both methods, written in matrix form.

I. The flow equations by Wegner [1]:

$$\frac{dH_{ij}}{dl} = [\eta, H_I]_{ij} + \frac{du_{ij}}{dl}\,\frac{H_{ij}}{u_{ij}}\ , \tag{234}$$

$$\eta_{ij} = \frac{1}{E_i - E_j}\left(-\frac{du_{ij}}{dl}\,\frac{H_{ij}}{u_{ij}}\right) \tag{235}$$

with

$$u_{ij} = \exp(-l\Delta_{ij}^2) \tag{236}$$

and

$$f_{ij} = u_{ij}\ . \tag{237}$$

II. The similarity unitary transformation by Glazek and Wilson [2]:

$$\frac{dH_{ij}}{d\lambda} = u_{ij}[\eta, H_I]_{ij} + r_{ij}\frac{du_{ij}}{d\lambda}\frac{H_{ij}}{u_{ij}} \ , \tag{238}$$

$$\eta_{ij} = \frac{r_{ij}}{E_i - E_j}\left([\eta, H_I]_{ij} - \frac{du_{ij}}{d\lambda}\frac{H_{ij}}{u_{ij}}\right) \tag{239}$$

and

$$f_{ij} = u_{ij}\exp(r_{ij}) \ . \tag{240}$$

Also the following transformation is used [2]:

$$\frac{dH_{ij}}{d\lambda} = u_{ij}[\eta, H_I]_{ij} + \frac{du_{ij}}{d\lambda}\frac{H_{ij}}{u_{ij}} \ , \tag{241}$$

$$\eta_{ij} = \frac{1}{E_i - E_j}\left(r_{ij}[\eta, H_I]_{ij} - \frac{du_{ij}}{d\lambda}\frac{H_{ij}}{u_{ij}}\right) \tag{242}$$

and

$$f_{ij} = u_{ij} \ , \tag{243}$$

where for both similarity transformations

$$u_{ij} = \theta(\lambda - |\Delta_{ij}|) \tag{244}$$

and

$$u_{ij} + r_{ij} = 1 \ . \tag{245}$$

Also other choices for the similarity function u_{ij} with the step behaviour are possible [2].

Remember, the function f_{ij} defines the solution for the leading order interaction term. The first order equations for H and η, written through the f-function are unique for both methods (**I.** and **II.**)

$$H^{(1)}_{I,ij}(l) = H^{(1)}_{I,ij}(l=0)\frac{f_{ij}(l)}{f_{ij}(l=0)} \ , \tag{246}$$

$$\eta^{(1)}_{ij}(l) = -\frac{1}{E_i - E_j}\frac{dH^{(1)}_{I,ij}}{dl} \tag{247}$$

with the connection given in eq. (233) in the renormalized values, and $dl \to d\lambda$ implied. The eqs. (246) and (247) will be exploited further for the calculations in the main text.

B Calculation of the commutator $[\eta^{(1)}(l), H_{ee\gamma}]$ in the electron-positron sector

Here we calculate the commutator $[\eta^{(1)}(l), H_{ee\gamma}]$ in the electron-positron sector. The leading order generator $\eta^{(1)}$ is:

$$\begin{aligned}\eta^{(1)}(l) &= \sum_{\lambda s_1 s_3}\int_{p_1 p_3 q}(\eta^*_{p_1p_3}(l)\varepsilon^i_\lambda \tilde{a}_q + \eta_{p_1p_3}(l)\varepsilon^{i*}_\lambda \tilde{a}^+_{-q})\,(\tilde{b}^+_{p_3}\tilde{b}_{p_1} + \tilde{b}^+_{p_3}\tilde{d}^+_{-p_1} + \tilde{d}_{-p_3}\tilde{b}_{p_1} + \tilde{d}_{-p_3}\tilde{d}^+_{-p_1})\\ &\quad\times\chi^+_{s_3}\Gamma^i_l(p_1,p_3,-q)\chi_{s_1}\,\delta_{q,-(p_1-p_3)}\ ,\end{aligned} \tag{248}$$

where

$$\eta_{p_1p_3}(l) = -\Delta_{p_1p_3}\cdot g_{p_1p_3} = \frac{1}{\Delta_{p_1p_3}}\cdot\frac{dg_{p_1p_3}}{dl}\ , \tag{249}$$

$\Delta_{p_1p_3} = p_1^- - p_3^- - (p_1-p_3)^-$, and the electron-photon coupling

$$\begin{aligned}H_{ee\gamma} &= \sum_{\lambda s_2 s_4}\int_{p_2p_4q'}(g^*_{p_2p_4}(l)\varepsilon^j_{\lambda'}\tilde{a}_{q'} + g_{p_2p_4}(l)\varepsilon^{j*}_{\lambda'}\tilde{a}^+_{-q'})\,(\tilde{b}^+_{p_4}\tilde{b}_{p_2} + \tilde{b}^+_{p_4}\tilde{d}^+_{-p_2} + \tilde{d}_{-p_4}\tilde{b}_{p_2} + \tilde{d}_{-p_4}\tilde{d}^+_{-p_2})\\ &\quad\times\chi^+_{s_4}\Gamma^i_l(p_2,p_4,-q')\chi_{s_2}\,\delta_{q',-(p_2-p_4)}\ ,\end{aligned} \tag{250}$$

where

$$\Gamma^i_l(p_1,p_2,q) = 2\frac{q^i}{q^+} - \frac{\sigma\cdot p_2^\perp - im}{p_2^+}\sigma^i - \sigma^i\frac{\sigma\cdot p_1^\perp + im}{p_1^+} \tag{251}$$

and the tilde-fields are defined in eq. (34). Further we use the identities for the polarisation vectors and spinors

$$\sum_\lambda \varepsilon^{i*}_\lambda\varepsilon^j_\lambda = \delta^{ij}\ ,\qquad \chi^+_s\chi_{s'} = \delta_{ss'}\ . \tag{252}$$

Using the commutation relations, eq. (26), and identities eq. (252) we have

$$\begin{aligned}[\eta^{(1)}(l), H_{ee\gamma}] &= \frac{1}{2}\left(-\eta_{p_1p_3}g^*_{p_2p_4}\frac{\theta(p_1^+-p_3^+)}{p_1^+-p_3^+} + \eta^*_{p_1p_3}g_{p_2p_4}\frac{\theta(p_3^+-p_1^+)}{p_3^+-p_1^+}\right)\\ &\quad\times{:}\,(-\tilde{b}^+_{p_3}\tilde{d}^+_{-p_2}\tilde{d}_{-p_4}\tilde{b}_{p_1} - \tilde{b}^+_{p_4}\tilde{d}^+_{-p_1}\tilde{d}_{-p_3}\tilde{b}_{p_2} + \tilde{b}^+_{p_3}\tilde{d}^+_{-p_1}\tilde{d}_{-p_4}\tilde{b}_{p_2} + \tilde{b}^+_{p_4}\tilde{d}^+_{-p_2}\tilde{d}_{-p_3}\tilde{b}_{p_1})\,{:}\\ &\quad\times(\chi^+_{s_3}\Gamma^i_l(p_1,p_3,p_1-p_3)\chi_{s_1})\,(\chi^+_{s_4}\Gamma^i_l(p_2,p_4,p_2-p_4)\chi_{s_2})\,\delta_{p_1+p_2,p_3+p_4}\ ,\end{aligned} \tag{253}$$

where the first two terms of the field operators contribute to the exchange channel, and the next two to the annihilation channel. We take into account both s- and t-channel terms to calculate the bound states. The : : stand for the normal ordering of the fermion operators and $(\frac{1}{2})$ is the symmetry factor. The sum over helicities s_i and the 3-dimensional integration over momenta p_i, $i = 1,..4$, according to eq. (35) is implied. We rewrite for both channels

$$[\eta, H_{ee\gamma}] = \begin{cases} M^{(ex)}_{2ij}(\frac{1}{2})\left\{\frac{\theta(p_1^+-p_3^+)}{(p_1^+-p_3^+)}\,(\eta_{p_1,p_3}g^*_{-p_4,-p_2} - \eta^*_{-p_4,-p_2}g_{p_1,p_3})\right.\\ \qquad\left. + \frac{\theta(-(p_1^+-p_3^+))}{-(p_1^+-p_3^+)}(\eta_{-p_4,-p_2}g^*_{p_1,p_3} - \eta^*_{p_1,p_3}g_{-p_4,-p_2})\right\}\\ \qquad\times\delta^{ij}\delta_{p_1+p_2,p_3+p_4}\,b^+_{p_3s_3}d^+_{p_4\bar{s}_4}d_{p_2\bar{s}_2}b_{p_1s_1}\\ -M^{(an)}_{2ij}(\frac{1}{2})\left\{\frac{\theta(p_1^++p_2^+)}{(p_1^++p_2^+)}\,(\eta_{p_1,-p_2}g^*_{-p_4,p_3} - \eta^*_{-p_4,p_3}g_{p_1,-p_2})\right.\\ \qquad\left. + \frac{\theta(-(p_1^++p_2^+))}{-(p_1^++p_2^+)}(\eta_{-p_4,p_3}g^*_{p_1,-p_2} - \eta^*_{p_1,-p_2}g_{-p_4,p_3})\right\}\\ \qquad\times\delta^{ij}\delta_{p_1+p_2,p_3+p_4}\,b^+_{p_3s_3}d^+_{p_4\bar{s}_4}d_{p_2\bar{s}_2}b_{p_1s_1}\end{cases} \tag{254}$$

where

$$
\begin{aligned}
M^{(ex)}_{2ij} &= (\chi^+_{s_3}\Gamma^i_l(p_1,p_3,p_1-p_3)\chi_{s_1})\,(\chi^+_{\bar{s}_2}\Gamma^j_l(-p_4,-p_2,-(p_1-p_3))\chi_{\bar{s}_4}) \\
M^{(an)}_{2ij} &= (\chi^+_{s_3}\Gamma^i_l(-p_4,p_3,-(p_1+p_2))\chi_{\bar{s}_4})\,(\chi^+_{\bar{s}_2}\Gamma^j_l(p_1,-p_2,p_1+p_2)\chi_{s_1})\,.
\end{aligned}
\tag{255}
$$

The first term in the exchange channel with $p_1^+ > p_3^+$ corresponds to the light-front time ordering $x_1^+ < x_3^+$ with the intermediate state $P_k^- = p_3^- + (p_1-p_3)^- + p_2^-$, the second term $p_1^+ < p_3^+$ and $x_1^+ > x_3^+$ has the intermediate state $P_k^- = p_1^- - (p_1-p_3)^- + p_4^-$. Both terms can be viewed as the retarded photon exchange. The same does hold for the annihilation channel.

Consider only real couplings and take into account the symmetry

$$
\eta_{-p_4,-p_2} = -\eta_{p_4,p_2}\,, \qquad g_{-p_4,-p_2} = g_{p_4,p_2}\,. \tag{256}
$$

Then $\langle p_3 s_3, p_4\bar{s}_4|\,[\eta^{(1)}, H_{ee\gamma}]\,|p_1 s_1, p_2\bar{s}_2\rangle$, the matrix element of the commutator between the free states of positronium in the exchange and annihilation channel, reads

$$
<[\eta^{(1)}, H_{ee\gamma}]>/\delta_{p_1+p_2,p_3+p_4} = \left\{ \begin{array}{l} M^{ex}_{2ii}\,\frac{1}{(p_1^+-p_3^+)}\,(\eta_{p_1,p_3}g_{p_4,p_2}+\eta_{p_4,p_2}g_{p_1,p_3}) \\ -M^{an}_{2ii}\,\frac{1}{(p_1^++p_2^+)}\,(\eta_{p_1,-p_2}g_{p_4,-p_3}+\eta_{p_4,-p_3}g_{p_1,-p_2}) \end{array} \right. . \tag{257}
$$

We rewrite this expression through the corresponding f-functions

$$
\begin{aligned}
\eta_{p_1,p_3}g_{p_4,p_2}+\eta_{p_4,p_2}g_{p_1,p_3} &= e^2\left[\frac{1}{\Delta_{p_1,p_3}}\frac{df_{p_1,p_3}(l)}{dl}f_{p_4,p_2}(l)+\frac{1}{\Delta_{p_4,p_2}}\frac{df_{p_4,p_2}(l)}{dl}f_{p_1,p_3}(l)\right] \\
\eta_{p_1,-p_2}g_{p_4,-p_3}+\eta_{p_4,-p_3}g_{p_1,-p_2} &= e^2\left[\frac{1}{\Delta_{p_1,-p_2}}\frac{df_{p_1,-p_2}(l)}{dl}f_{p_4,-p_3}(l)+\frac{1}{\Delta_{p_4,-p_3}}\frac{df_{p_4,-p_3}(l)}{dl}f_{p_1,-p_2}(l)\right]
\end{aligned}
\tag{258}
$$

with $\Delta_{p_1,p_2} = p_1^- - p_2^- - (p_1-p_2)^-$. As we have mentioned in Appendix A this form in terms of the f-function is universal for all unitary transformations. We exploit further this expression by specifying the f-function to compare the effective interactions in different renormalization schemes (see Appendix C).

We calculate the matrix elements M_{2ii}, eq. (188), for both channels. Here we follow the notations introduced in [6].

We make use of the identities

$$
\chi^+_s\sigma^i\sigma^j\chi_s = \delta^{ij} + is\varepsilon^{ij}\,, \qquad \chi^+_s\sigma^j\sigma^i\chi_s = \delta^{ij} + i\bar{s}\varepsilon^{ij} \tag{259}
$$

with $\bar{s} = -s$ and $\chi^+_s\chi_{s'} = \delta_{ss'}$; also of

$$
\chi^+_{\bar{s}}\sigma^i\chi_s = -\sqrt{2}s\varepsilon^i_s\,, \qquad \chi^+_s\sigma^i\chi_{\bar{s}} = -\sqrt{2}s\varepsilon^{i*}_s \tag{260}
$$

with $\varepsilon^*_s = -\varepsilon_{\bar{s}}$ and $\varepsilon^i_s\varepsilon^i_{s'} = -\delta_{s\bar{s}'}$.

We use the standard light-front frame, fig. (8),

$$
\begin{aligned}
p_1 &= (xP^+, xP^\perp + k_\perp)\,, \qquad p_2 = ((1-x)P^+, (1-x)P^\perp - k_\perp)\,, \\
p_3 &= (x'P^+, x'P^\perp + k'_\perp)\,, \qquad p_4 = ((1-x')P^+, (1-x')P^\perp - k'_\perp)\,,
\end{aligned}
\tag{261}
$$

where $P = (P^+, P^\perp)$ is the positronium momentum.

Then, to calculate the matrix element M_{2ii} in the **exchange channel**, we find

$$\begin{aligned} P^+[\chi^+_{s_3}\Gamma^i(p_1,p_3,p_1-p_3)\chi_{s_1}] &= \chi^+_{s_3}\left[2\frac{(k_\perp-k'_\perp)^i}{(x-x')}-\frac{\sigma\cdot k'_\perp}{x'}\sigma^i+\sigma^i\frac{\sigma\cdot k_\perp}{x}+im\frac{x-x'}{xx'}\sigma^i\right]\chi_{s_1} \\ &= T^i_2\delta_{s_1s_3}+im\frac{x-x'}{xx'}(-\sqrt{2})s_1\varepsilon^i_{s_1}\delta_{s_1\bar{s}_3}\ , \end{aligned} \tag{262}$$

and

$$\begin{aligned} &P^+[\chi^+_{\bar{s}_2}\Gamma^i(-p_4,-p_2,-(p_4-p_2))\chi_{\bar{s}_4}] \\ &\qquad = \chi^+_{\bar{s}_2}\left[2\frac{(k_\perp-k'_\perp)^i}{x-x'}+\left(\frac{\sigma\cdot k_\perp}{1-x}\sigma^i+\sigma^i\frac{\sigma\cdot k'_\perp}{1-x'}\right)-im\frac{x-x'}{(1-x)(1-x')}\sigma^i\right]\chi_{\bar{s}_4} \\ &\qquad = -\left[T^i_1\delta_{s_2s_4}+im\frac{x-x'}{(1-x)(1-x')}(-\sqrt{2})s_2\varepsilon^i_{s_2}\delta_{s_2\bar{s}_4}\right]\ , \end{aligned} \tag{263}$$

where we have introduced

$$\begin{aligned} T^i_1 &\equiv -\left[2\frac{(k_\perp-k'_\perp)^i}{x-x'}+\frac{k^i_\perp(s_2)}{(1-x)}+\frac{k'^i_\perp(\bar{s}_2)}{(1-x')}\right] \\ T^i_2 &\equiv 2\frac{(k_\perp-k'_\perp)^i}{x-x'}-\frac{k^i_\perp(s_1)}{x}-\frac{k'^i_\perp(\bar{s}_1)}{x'} \end{aligned} \tag{264}$$

and

$$k^i_\perp(s)\equiv k^i_\perp+is\,\varepsilon_{ij}\,k^j_\perp\ . \tag{265}$$

Finaly we result

$$\begin{aligned} P^{+2}\,M^{(ex)}_{2ii} = -\Bigg\{&\delta_{s_1s_3}\delta_{s_2s_4}T^\perp_1\cdot T^\perp_2-\delta_{s_1\bar{s}_2}\delta_{s_1\bar{s}_3}\delta_{s_2\bar{s}_4}2m^2\frac{(x-x')^2}{xx'(1-x)(1-x')} \\ &+im\sqrt{2}(x'-x)\left[\delta_{s_1\bar{s}_3}\delta_{s_2s_4}\frac{s_1}{xx'}T^\perp_1\cdot\varepsilon^\perp_{s_1}+\delta_{s_1s_3}\delta_{s_2\bar{s}_4}\frac{s_2}{(1-x)(1-x')}T^\perp_2\cdot\varepsilon^\perp_{s_2}\right]\Bigg\}\ . \end{aligned} \tag{266}$$

Whereas in the **annihilation channel** we calculate

$$\begin{aligned} P^+[\chi^+_{s_3}\Gamma^i(-p_4,p_3,-(p_1+p_2))\chi_{\bar{s}_4}] &= \chi^+_{s_3}\left[-\frac{\sigma\cdot k'_\perp}{x'}\sigma^i+\sigma^i\frac{\sigma\cdot k'_\perp}{1-x'}+im\frac{1}{x'(1-x')}\sigma^i\right]\chi_{\bar{s}_4} \\ &= T^i_3\delta_{s_3\bar{s}_4}+im\frac{1}{x'(1-x')}(-\sqrt{2})s_4\varepsilon^{i*}_{s_4}\delta_{s_3s_4} \end{aligned} \tag{267}$$

and

$$\begin{aligned} P^+[\chi^+_{\bar{s}_2}\Gamma^i(p_1,-p_2,p_1+p_2)\chi_{s_1}] &= \chi^+_{\bar{s}_2}\left[\frac{\sigma\cdot k_\perp}{1-x}\sigma^i-\sigma^i\frac{\sigma\cdot k_\perp}{x}-im\frac{1}{x(1-x)}\sigma^i\right]\chi_{s_1} \\ &= T^i_4\delta_{s_1\bar{s}_2}-im\frac{1}{x(1-x)}(-\sqrt{2})s_1\varepsilon^i_{s_1}\delta_{s_1s_2}\ , \end{aligned} \tag{268}$$

where we have introduced

$$\begin{aligned} T^i_3 &\equiv -\frac{k'^i_\perp(\bar{s}_3)}{x'}+\frac{k'^i_\perp(s_3)}{1-x'} \\ T^i_4 &\equiv \frac{k^i_\perp(\bar{s}_1)}{1-x}-\frac{k^i_\perp(s_1)}{x}\ . \end{aligned} \tag{269}$$

We finally have

$$P^{+2}\, M^{(an)}_{2ii} = \delta_{s_1\bar{s}_2}\delta_{s_3\bar{s}_4} T_3^{\perp}\cdot T_4^{\perp} + \delta_{s_1 s_2}\delta_{s_3 s_4}\delta_{s_1 s_3} 2m^2 \frac{1}{xx'(1-x)(1-x')} \\ + im\sqrt{2}\left[\delta_{s_3\bar{s}_4}\delta_{s_1 s_2}\frac{s_1}{x(1-x)} T_3^{\perp}\cdot\varepsilon^{\perp}_{s_1} - \delta_{s_3 s_4}\delta_{s_1\bar{s}_2}\frac{s_3}{x'(1-x')} T_4^{\perp}\cdot\varepsilon^{\perp *}_{s_4}\right] . \tag{270}$$

C Fermion and photon self energy terms

We calculate here the fermion and photon self energy terms, arising from the second order commutator $[\eta^{(1)}, H_{ee\gamma}]$.

I. We first derive the **electron self energy** terms. Making use of the expressions for the generator of the unitary transformation $\eta^{(1)}$ defined in eq. (118) and of $H_{ee\gamma}$, eq. (30), we obtain the following expression for the commutator in the electron self energy sector

$$\frac{1}{2}(\eta_{p_1p_2}g_{p_2p_1} - \eta_{p_2p_1}g_{p_1p_2})\left[\theta(p_1^+)\frac{\theta(p_2^+ - p_1^+)}{p_2^+ - p_1^+}\theta(p_2^+)\, b^+_{p_2} b_{p_2}\chi^+_{s_2}\chi_{s_2}\right. \\ \left. -\theta(p_2^+)\frac{\theta(p_1^+ - p_2^+)}{p_1^+ - p_2^+}\theta(p_1^+)\, b^+_{p_1} b_{p_1}\chi^+_{s_1}\chi_{s_1}\right] M_{2ij}(p_1,p_2)\delta^{ij} , \tag{271}$$

where

$$M_{2ij}(p_1,p_2) = \Gamma^i(p_1,p_2,p_1-p_2)\Gamma^j(p_2,p_1,p_2-p_1) \tag{272}$$

and the momentum integration over p_1, p_2 is implied; $1/2$ stands as the symmetry factor. The matrix element of the commutator between the free fermion states is

$$< p_1,s_1|[\eta^{(1)}, H_{ee\gamma}]|p_1,s_1>_{selfenergy} \\ = -\int_{p_2}(\eta_{p_1p_2}g_{p_2p_1} - \eta_{p_2p_1}g_{p_1p_2})\,\theta(p_2^+)\frac{\theta(p_1^+ - p_2^+)}{p_1^+ - p_2^+}\, M_{2ii}(p_1,p_2) , \tag{273}$$

where the integration $\int_p$ is defined in eq. (35). We use the expression for the generator η through the coupling, namely

$$\eta_{p_1p_2}g_{p_2p_1} - \eta_{p_2p_1}g_{p_1p_2} = \frac{1}{\Delta_{p_1p_2}}\left(g_{p_1p_2}\frac{dg_{p_2p_1}}{dl} + g_{p_2p_1}\frac{dg_{p_1p_2}}{dl}\right) . \tag{274}$$

Change of the variables according to

$$\begin{aligned} p_1 &= p \\ p_2 &= p_k \\ p_1 - p_2 &= k \end{aligned} \tag{275}$$

brings the integral in eq. (273) to the standard form of loop integration

$$-\int_k(\eta_{p,p-k}g_{p-k,p} - \eta_{p-k,p}g_{p,p-k})\,\theta(p^+ - k^+)\frac{\theta(k^+)}{k^+}\, M_{2ii}(p,p-k) . \tag{276}$$

According to eq. (59), the integral $\int_{l_\lambda}^{l_\Lambda}$ of the commutator $[\eta^{(1)}, H_{ee\gamma}]$ defines the difference between the energies (or energy corrections) $\delta p^-_{1\lambda} - \delta p^-_{1\Lambda}$. Making use of

$$\int_{l_\Lambda}^{l_\lambda} dl'(\eta_{p_1p_2}g_{p_2p_1} - \eta_{p_2p_1}g_{p_1p_2}) = \frac{1}{p_1^- - p_2^- - (p_1-p_2)^-}\,(g_{p_1,p_2,\Lambda}g_{p_2,p_1,\lambda} - g_{p_1,p_2,\lambda}g_{p_2,p_1,\Lambda}) \tag{277}$$

we have the following explicit expression:

$$\delta p^-_{1\lambda} - \delta p^-_{1\Lambda} = e^2 \int \frac{d^2k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+}\theta(p^+ - k^+) \frac{(-1)}{p^- - k^- - (p-k)^-} \tag{278}$$
$$\times\Gamma^i(p-k,p,-k)\Gamma^i(p,p-k,k) \left[\exp\left\{-2\left(\frac{\Delta_{p,p-k}}{\lambda}\right)^2\right\} - \exp\left\{-2\left(\frac{\Delta_{p,p-k}}{\Lambda}\right)^2\right\}\right] ,$$

where the solution for the $ee\gamma$-coupling constant was used. Therefore the electron energy correction corresponding to the first diagram, fig. (6), is

$$\begin{aligned}\delta p^-_{1\lambda} &= e^2 \int \frac{d^2k^\perp dk^+}{2(2\pi)^3} \frac{\theta(k^+)}{k^+}\theta(p^+ - k^+) \\ &\times\Gamma^i(p-k,p,-k)\Gamma^i(p,p-k,k) \frac{1}{p^- - k^- - (p-k)^-} \times(-R) ,\end{aligned} \tag{279}$$

where we have introduced the regulator R, defining the cutoff condition (see main text),

$$R = \exp\left\{-2\left(\frac{\Delta_{p,k}}{\lambda}\right)^2\right\} \tag{280}$$

(note that $\Delta_{p,k} = \Delta_{p,p-k}$). To perform the integration over $k = (k^+, k^\perp)$ explicitly, choose the parametrization

$$\begin{aligned}\frac{k^+}{p^+} &= x \\ k &= (xp^+, xp^\perp + \kappa^\perp) ,\end{aligned} \tag{281}$$

where $p = (p^+, p^\perp)$ is the external electron momentum. Then the terms occuring in $\delta p^-_{1\lambda}$ are rewritten in the form

$$\begin{aligned}\Gamma^i(p-k,p,-k)\Gamma^i(p,p-k,k) &= \tfrac{1}{(p^+)^2(1-x)^2}\left(\left(4\tfrac{1}{x^2} - 4\tfrac{1}{x} + 2\right)\kappa_\perp^2 + 2m^2x^2\right) \\ \Delta_{p,p-k} = p^- - k^- - (p-k)^- &= \tfrac{1}{p^+x(1-x)}(x(1-x)p^2 - \kappa_\perp^2 - xm^2) = \tfrac{\tilde\Delta_{p,p-k}}{p^+} \quad .\end{aligned} \tag{282}$$

Therefore the integral for the electron energy correction corresponding to the first diagram of fig. (6) takes the form

$$\begin{aligned}p^+\delta p^-_{1\lambda} &= -\frac{e^2}{8\pi^2}\int_0^1 dx \int d\kappa_\perp^2 \frac{(\frac{2}{x^2} - \frac{2}{x} + 1)\kappa_\perp^2 + m^2x^2}{(1-x)(\kappa_\perp^2 + f(x))} \times(-R) \\ &= -\frac{e^2}{8\pi^2}\int_0^1 dx \int d\kappa_\perp^2 \\ &\quad\times\left[\frac{p^2 - m^2}{\kappa_\perp^2 + f(x)}\left(\frac{2}{[x]} - 2 + x\right) - \frac{2m^2}{\kappa_\perp^2 + f(x)} + \left(\frac{2}{[x]^2} + \frac{1}{[1-x]}\right)\right]\times(-R) ,\end{aligned} \tag{283}$$

where

$$f(x) = xm^2 - x(1-x)p^2 . \tag{284}$$

In the last integral the principal value prescription for $\frac{1}{[x]}$ as $x \to 0$ was introduced (see main text), to regularize the IR divergencies present in the longitudinal direction.

We thus have derived the expression for the energy correction which has been used in the main text.

II. We repeat the same procedure for the **photon self energy**. The second order commutator $[\eta^{(1)}, H_{ee\gamma}]$ gives the following expression in the photon self energy sector

$$\frac{1}{2}(\eta_{p_1p_2}g_{p_2p_1} - \eta_{p_2p_1}g_{p_1p_2}) \cdot \Big[\theta(p_1^+)\theta(-p_2^+)\frac{\theta(p_1^+ - p_2^+)}{(p_1^+ - p_2^+)}\, a^+_{-q}a_{-q}\varepsilon^{i*}_\lambda\varepsilon^j_\lambda \tag{285}$$
$$-\theta(-p_1^+)\theta(p_2^+)\frac{\theta(p_2^+ - p_1^+)}{(p_2^+ - p_1^+)}\, a^+_q a_q\varepsilon^i_\lambda\varepsilon^{j*}_\lambda\Big] \cdot TrM_{2ij}(p_1,p_2)\,\delta_{q,-(p_1-p_2)}\ ,$$

where $M_{2ij}(p_1,p_2)$ is defined in eq. (272) and the trace acts in spin space; the integration over the momenta q, p_1 and p_2 is implied. The matrix element between the free photon states reads

$$< q,\lambda|[\eta^{(1)}, H_{ee\gamma}]|q,\lambda >_{selfenergy} \delta_{ij} \tag{286}$$
$$= -\frac{1}{q^+}\int_{p_1,p_2}(\eta_{p_1p_2}g_{p_2p_1} - \eta p_2p_1 g_{p_1p_2})\,\theta(-p_1^+)\theta(p_2^+)\,TrM_{2ij}(p_1,p_2)\,\delta_{q,-(p_1-p_2)}\ ,$$

that can be rewritten after the change of coordinates according to

$$\begin{aligned} p_1 &= -k \\ p_2 &= -(k-q) \\ p_2 - p_1 &= q \end{aligned} \tag{287}$$

in the following way

$$\frac{1}{q^+}\int_k(\eta_{k,k-q}g_{k-q,k} - \eta_{k-q,k}g_{k,k-q})\,\theta(k^+)\theta(q^+ - k^+)\,TrM_{2ij}(k,k-q)\ , \tag{288}$$

where the symmetry

$$\begin{aligned} \eta_{-p_1,-p_2} &= -\eta_{p_1,p_2} \\ g_{-p_1,-p_2} &= g_{p_1,p_2} \end{aligned} \tag{289}$$

has been used. The integration of the commutator over l in the flow equation gives rise to

$$(\delta q^-_{1\lambda} - \delta q^-_{1\Lambda})\delta^{ij} = \frac{1}{q^+}e^2\int\frac{d^2k^\perp dk^+}{2(2\pi)^3}\theta(k^+)\theta(q^+-k^+)\,\frac{(-1)}{q^- - k^- - (q-k)^-} \tag{290}$$
$$\times Tr\left(\Gamma^i(k,k-q,q)\Gamma^j(k-q,k,-q)\right)\left[\exp\left\{-2\left(\frac{\Delta_{q,q-k}}{\lambda}\right)^2\right\} - \exp\left\{-2\left(\frac{\Delta_{q,q-k}}{\Lambda}\right)^2\right\}\right].$$

This means for the photon energy correction

$$\delta q^-_{1\lambda}\delta^{ij} = \frac{1}{q^+}e^2\int\frac{d^2k^\perp dk^+}{2(2\pi)^3}\theta(k^+)\theta(q^+-k^+) \tag{291}$$
$$\times Tr\left(\Gamma^i(k,k-q,q)\Gamma^j(k-q,k,-q)\right)\frac{1}{q^- - k^- - (q-k)^-}\ \times(-R)\ ,$$

where the regulator R

$$R = \exp\left\{-2\left(\frac{\Delta_{q,k}}{\lambda}\right)^2\right\} \tag{292}$$

has been introduced. Define the new set of coordinates

$$\begin{aligned} \frac{(q-k)^+}{q^+} &= x \\ k &= ((1-x)q^+, (1-x)q^\perp + \kappa^\perp) \\ q-k &= (xq^+, xq^\perp - \kappa^\perp)\ , \end{aligned} \tag{293}$$

where $q=(q^+,q^\perp)$ is the photon momentum. Then the terms present in $\delta q^-_{1\lambda}$ are

$$\Gamma^i(k,k-q,q)\Gamma^i(k-q,k,-q)=\tfrac{2}{(q^+)^2x(1-x)^2}\left(\left(2x-2+\tfrac{1}{x}\right)\kappa^{\perp 2}+\tfrac{m^2}{x}\right)$$
$$\Delta_{k-q,k}=q^- -k^- -(q-k)^- =-\tfrac{\kappa^{\perp 2}+m^2}{q^+x(1-x)}+\tfrac{q^2}{q^+}=\tfrac{\tilde{\Delta}_{k-q,k}}{q^+} \quad . \tag{294}$$

The integral for the photon energy correction corresponding to the first diagram of fig. (7) takes the form

$$q^+\delta q^-_{1\lambda}=-\frac{e^2}{8\pi^2}\int_0^1 dx\int d\kappa_\perp^2\,\frac{(2x-2+\frac{1}{x})\kappa_\perp^2+\frac{m^2}{x}}{(1-x)(\kappa_\perp^2+f(x))}\times(-R) \tag{295}$$
$$=-\frac{e^2}{8\pi^2}\int_0^1 dx\int d\kappa_\perp^2\left\{\frac{q^2}{\kappa_\perp^2+f(x)}\left(2x^2-2x+1\right)+\frac{2m^2}{[1-x]}+\left(-2+\frac{1}{[x][1-x]}\right)\right\}\times(-R)$$

with

$$f(x)=m^2-q^2x(1-x)\ , \tag{296}$$

and the principal value prescription, denoted by '[ ]', introduced to regularize the IR divergencies.

This is the form of the photon correction used in the main text.

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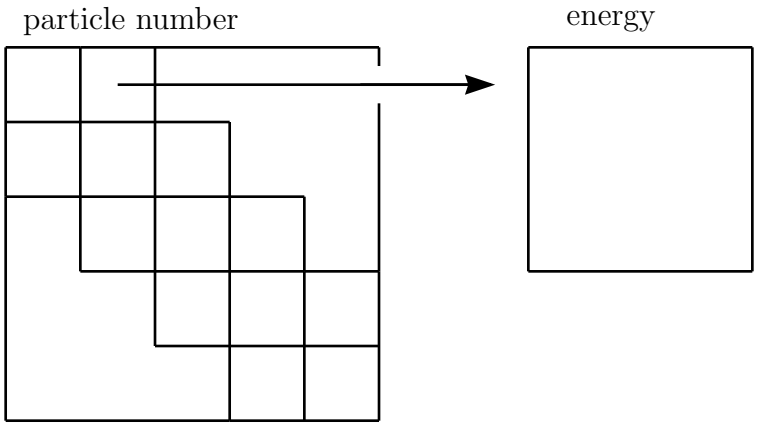

Figure 1: Pentadiagonal form of field theoretical Hamiltonian in 'particle number' space (the case of one-body Hamiltonian). Not squared region corresponds to zero matrix elements.

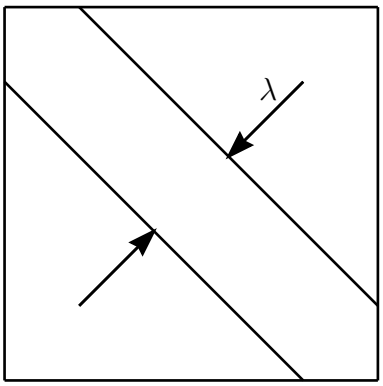

Figure 2: Similarity renormalization (SR) in 'energy space' results to the band-diagonal form for effective Hamiltonian, where all matrix elements are sqeezed in the energy band $|E_i - E_j| < \lambda$.

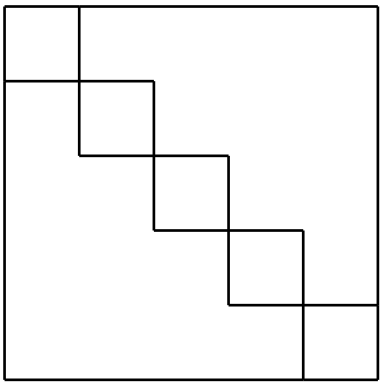

Figure 3: Similarity renormalization (SR) in 'particle number space' results to the block-diagonal form for effective Hamiltonian, where each block (sector) conserves the number of particles.

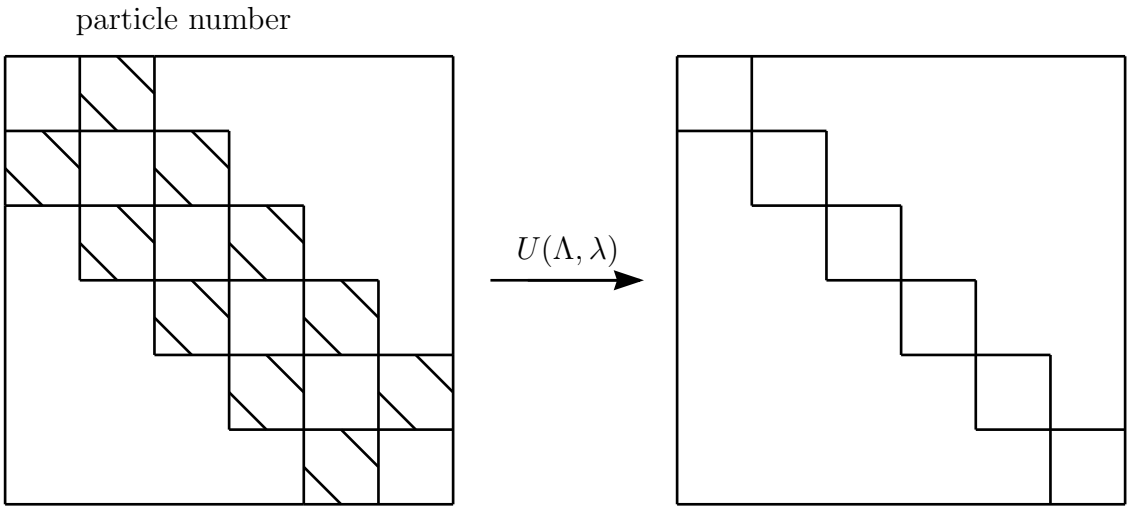

Figure 4: Modified similarity renormalization (MSR) of field theoretical Hamiltonians combines both similarity renormalization in 'particle number' and 'energy' representations. For the finite value of λ (after the unitary transformation $U(\Lambda \to \infty, \lambda)$ is performed) the matrix elements of 'particle number changing' sectors are sqeezed in the energy band $|E_i - E_j| < \lambda$ on the left hand side picture and are eliminated completly as $\lambda \to 0$ (that corresponds to $U(\Lambda \to \infty, \lambda \to 0)$) on the right hand side picture. One ends up with the block-diagonal in 'particle number space' renormalized effective Hamiltonian.

Table 1: The renormalized to the second order effective light front QED Hamiltonian matrix in the Fock space. The matrix elements of the 'diagonal' (Fock state conserving) sectors exist for any energy differences; the 'rest' (Fock state changing) sectors are sqeezed in the energy band $\Delta_{p_i p_f} = |\sum p_i^- - \sum p_f^-| < \lambda$; black dots correspond to the zero matrix elements to the order $O(e^2)$. Instantaneous diagramms are not included.

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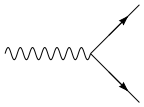
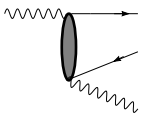
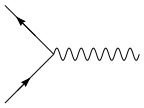
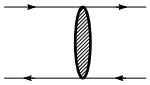
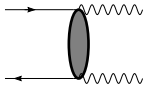
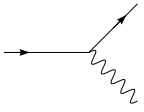
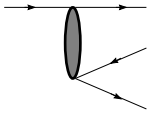
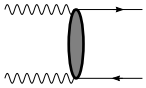
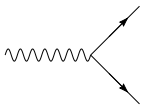
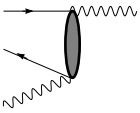
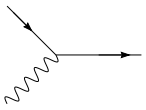
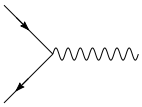
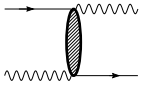
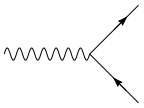
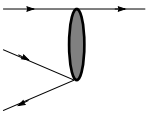
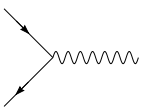
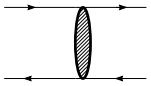
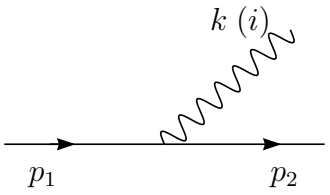
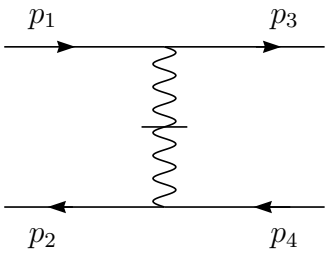
$\|\gamma>$					
$\|e\bar{e}>$					
$\|\gamma\gamma>$					
$\|e\bar{e}\gamma>$					
$\|e\bar{e}e\bar{e}>$					



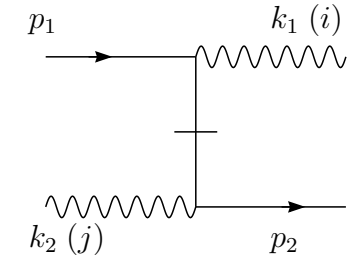
$$-e_\lambda \exp\left\{-\frac{\Delta^2_{p_1p_2}}{\lambda^2}\right\}\chi_2^+\Gamma^i_\lambda(p_1,p_2,k)\chi_1\varepsilon^{i*}$$

$$\Gamma^i_\lambda(p_1,p_2,k) = 2\frac{k^i}{[k^+]} - \frac{\sigma\cdot p_2^\perp - im_\lambda}{[p_2^+]}\sigma^i - \sigma^i\frac{\sigma\cdot p_1^\perp + im_\lambda}{[p_1^+]}$$

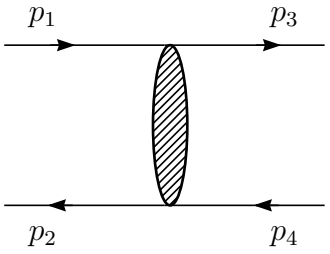
$$i = 1,2$$



$$e^2_\lambda\ \chi_3^+\chi_4^+\frac{4}{[p_1^+ - p_3^+]^2}\chi_1\chi_{\bar 2}$$



$$e^2_\lambda\chi_2^+\frac{\sigma^j\sigma^i}{[p_1^+ - k_1^+]}\chi_1\varepsilon^{i*}\varepsilon^j$$

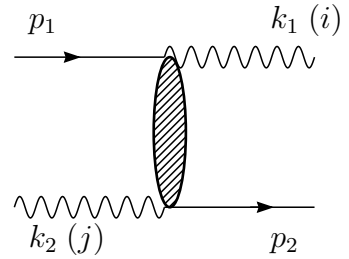


$$-e^2_\lambda M_{2ij,\lambda}\,\delta^{ij}\frac{1}{[p_1^+ - p_3^+]}$$

$$\times\left(\frac{\Delta_{p_1p_3\lambda}+\Delta_{p_4p_2\lambda}}{\Delta^2_{p_1p_3\lambda}+\Delta^2_{p_4p_2\lambda}}\right)\left(1-\exp\left\{-\frac{\Delta^2_{p_1p_3\lambda}+\Delta^2_{p_4p_2\lambda}}{\lambda^2}\right\}\right)$$

$$M_{2ij,\lambda} = \Big(\chi_3^+\Gamma^i_\lambda(p_1,p_3,p_1-p_3)\chi_1\Big)$$

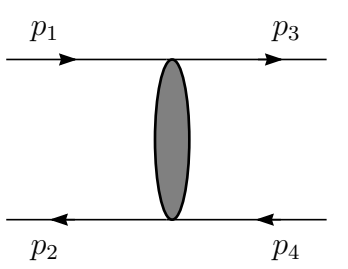
$$\times\Big(\chi_{\bar 2}^+\Gamma^j_\lambda(-p_4,-p_2,-(p_1-p_3))\chi_{\bar 4}\Big)$$



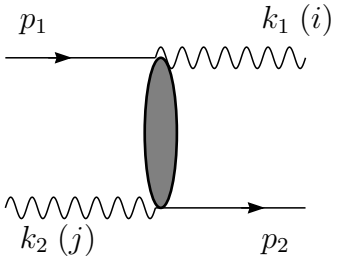
$$e^2_\lambda\widetilde{M}_{2ij,\lambda}\,\varepsilon^{i*}\varepsilon^j$$

$$\times\left(\frac{\Delta_{p_1k_1\lambda}+\Delta_{p_2k_2\lambda}}{\Delta^2_{p_1k_1\lambda}+\Delta^2_{p_2k_2\lambda}}\right)\left(1-\exp\left\{-\frac{\Delta^2_{p_1k_1\lambda}+\Delta^2_{p_2k_2\lambda}}{\lambda^2}\right\}\right)$$

$$\widetilde{M}_{2ij,\lambda} = \chi_2^+\Gamma^i_\lambda(p_1,p_1-k_1,k_1)\,\Gamma^j_\lambda(p_1-k_1,p_2,k_2)\chi_1$$



$$-e_\lambda^2 \exp\left\{-\frac{\Delta^2_{p_i p_f}}{\lambda^2}\right\} M_{2ij,\lambda}\, \delta^{ij} \frac{1}{[p_1^+ - p_3^+]}$$
$$\times \frac{1}{2}\left(\frac{1}{\Delta_{p_1 p_3 \lambda}} + \frac{1}{\Delta_{p_4 p_2 \lambda}}\right)\left(1 - \exp\left\{-2\,\frac{\Delta_{p_1 p_3 \lambda}\cdot\Delta_{p_4 p_2 \lambda}}{\lambda^2}\right\}\right)$$
$$M_{2ij,\lambda} = \left(\chi_2^+ \Gamma^i_\lambda(p_1, p_2, p_1 - p_2)\chi_1\right)$$
$$\times\left(\chi_4^+ \Gamma^j_\lambda(p_3, p_4, -(p_1 - p_2))\chi_3\right)$$



$$e_\lambda^2 \exp\left\{-\frac{\Delta^2_{p_i p_f}}{\lambda^2}\right\} \widetilde{M}_{2ij,\lambda}\, \varepsilon^{i*}\varepsilon^j$$
$$\times \frac{1}{2}\left(\frac{1}{\Delta_{p_1 k_1 \lambda}} + \frac{1}{\Delta_{p_2 k_2 \lambda}}\right)\left(1 - \exp\left\{-2\,\frac{\Delta_{p_1 k_1 \lambda}\cdot\Delta_{p_2 k_2 \lambda}}{\lambda^2}\right\}\right)$$
$$\widetilde{M}_{2ij,\lambda} = \chi_2^+ \Gamma^i_\lambda(p_1, p_1 - k_1, k_1)\, \Gamma^j_\lambda(p_1 - k_1, p_2, k_2)\chi_1$$

Figure 5: The matrix elements of the renormalized to the second order effective Hamitonian together with corresponding diagrams. The diagrams $2-5$ belong to the 'diagonal' sector; the $1, 6, 7$ correspond to the 'rest' sector (the $6, 7$ diagrams are drawn schematically, namely the corresponding momentum change must be performed to get the real 'rest' diagrams, depicted in Table 1.) The photon momenta are x^+-ordered, from left to right. The similarity function is chosen $f_{p_i p_f,\lambda} = \exp(-\Delta^2_{p_i p_f \lambda}/\lambda^2)$, where $\Delta_{p_i p_f \lambda} = \Sigma p_i^- - \Sigma p_f^-$ (the index 'i' denotes initial and 'f' final states) and $\Delta_{p_1 p_2 \lambda} = p_1^- - p_2^- - (p_1 - p_2)^-$, $p^- = (p_\perp^2 + m_\lambda^2)/p^+$.

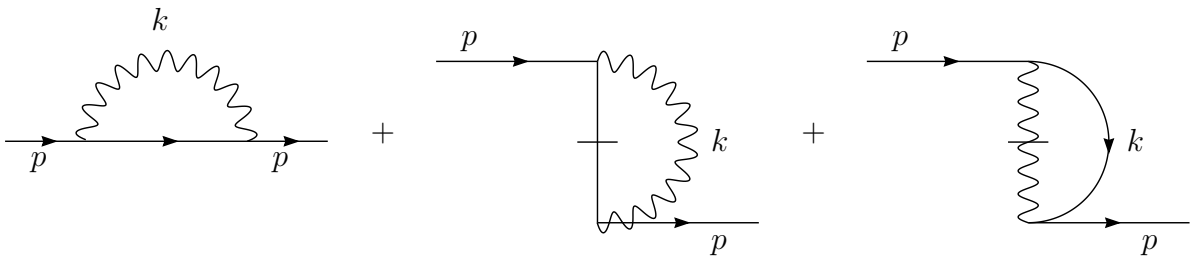


Figure 6: Electron self energy: the first diagram corresponds to the commutator term $[\eta^{(1)}, H_{ee\gamma}]$ in the electron self energy sector, next two diagrams arise from the normal ordering of instantaneous interactions.

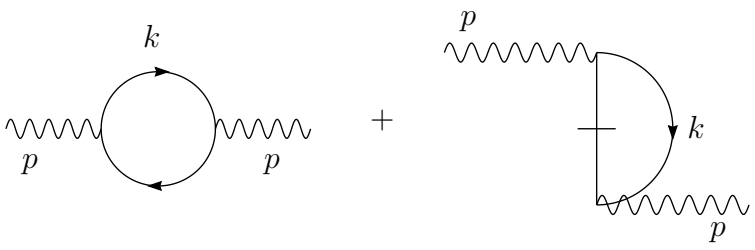


Figure 7: Photon self energy: the first diagram comes from the commutator $[\eta^{(1)}, H_{ee\gamma}]$ in the photon self energy sector, the second one from the normal ordering of the instantaneous interaction.

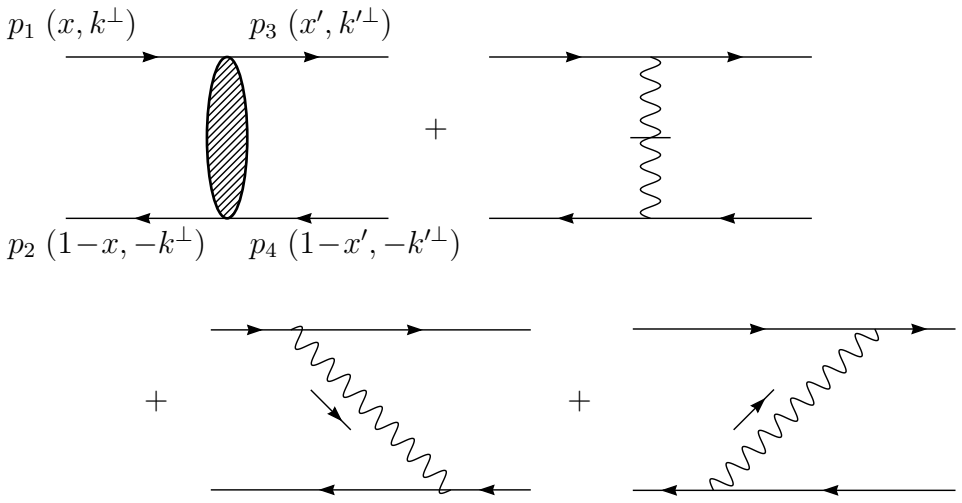


Figure 8: The renormalized to the second order effective electron-positron interaction in the exchange channel; diagrams correspond to generated, instantaneous interactions and two perturbative photon exchanges with respect to different time ordering.